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Methods of proving uniqueness of stationary distributions for stochastic PDEs

Paul Robert Horridge

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Mathematics Institute, University of Warwick, Coventry

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Declaration

The material contained in this thesis is original except where otherwise stated, and has not been previously submitted for a degree at any university.

Introduction

In this thesis, we consider solutions $u = u(t, x)$ for $t \geq 0$ and $x \in \mathbb{R}$, in time and one space dimension, of stochastic PDEs of the form

$$\partial_t u = \Delta u + a(u) + b(u)\, dW \tag{1}$$

where W is space-time white noise. The area is surveyed in Pardoux [25], and an introduction to the concepts of white noise and SPDE solutions can be found in Walsh [32].

We restrict attention to the case where a and b are continuous, although when $b = 1$, the equation has been studied where a is not even locally bounded (for example, Gyöngy and Pardoux [9]), and we suppose that solutions are started from some continuous initial condition f, and are continuous in (t, x).

Our results concern the uniqueness of a stationary distribution, i.e. a distribution μ on the set of initial conditions such that if u is a solution started from a random initial condition distributed according to μ, then for each $t \geq 0$, $u(t)$ also has distribution μ. We demonstrate how to use two different techniques (coupling and duality) to prove uniqueness for two separate classes of equations of (1).

In Chapter 1, we review the background we need for the later chapters. Most of the results in this chapter exist in some form in the literature, but we generalise several results — notably the comparison result in Section 1.2 — so they are suitable for our needs. Chapters 2 and 3 contain the main original work.

In Chapter 2, we suppose that a is nonincreasing and satisfies additional conditions in order for us to be able to control moments of solutions, that $b \in [1/L, L]$ for some $L \geq 0$ and that a and b are both Lipschitz. We then show that if f and g satisfy a certain growth condition, we may construct a pair of solutions u and v such that for large time, u and v agree on a compact interval in space with arbitrarily high probability. This coupling will then be used to prove uniqueness of the stationary distribution.

In Chapter 3, we consider the specific equation where $a(u) = \theta u - u^2$ and $b(u) = \sqrt{2u}$, and $\theta > 0$ is large enough to ensure that the process doesn't die out. In this special case, we can derive a duality relation for this equation and use it to show that the Laplace transforms of a nontrivial stationary distribution satisfying a certain mixing property are determined by the death probabilities of an independent process.

Coupling is a key tool in determining uniqueness of a stationary distribution for particle systems (Liggett [14]), and more general Markov processes, and has just recently been applied to SPDEs (Mueller claims the first application of a coupling method to SPDEs in his 1993 paper [16]). Duality is also a well-known technique in the area of particle systems (Liggett [14]). It has also been applied to prove uniqueness of solutions to SPDEs — for example, Athreya and Tribe [1] derive a duality relation in terms of a dual process given by a branching particle system with branching and death rates given by the power series representations of the coefficients of their SPDE, and in several papers of Mytnik ([19], [20], [21], [22]) a duality technique is used to prove uniqueness of the solution of various martingale problems.

Chapter 1

Background and basic results

1.1 Introduction to stochastic PDEs

We consider solutions $u = u(t, x)$ in one space dimension of stochastic PDEs of the form

$$\left.\begin{aligned} \partial_t u &= \Delta u + a(u) + b(u)\, dW \\ u(0) &= f \end{aligned}\right\} \tag{1.1}$$

where a and b are continuous, real-valued functions, f is some continuous deterministic initial condition, and W is space-time white noise.

First, we need to define the function space C_{tem}, as in Shiga [29], as follows. For each $\lambda > 0$ and $f \in C(\mathbb{R})$ we set

$$||f||_\lambda = \sup_{x \in \mathbb{R}} e^{-\lambda|x|}|f(x)|.$$

Now we set

$$C_{tem} = \{f \in C(\mathbb{R}) : ||f||_\lambda < \infty \text{ for all } \lambda > 0\}.$$

Thus C_{tem} consists of functions in $C(\mathbb{R})$ growing no faster than any exponential growth function. We equip C_{tem} with the topology generated by the norms $\|\ \|_\lambda$. This is metrisable by

$$d_{tem}(f,g) = \sum_{\lambda \in 1/\mathbb{N}} 2^{-1/\lambda} \left(\|f - g\|_\lambda \wedge 1 \right).$$

Note that the embedding $C_b \hookrightarrow C_{tem}$ of bounded continuous functions is continuous. We can also put a metric on the space $C([0,\infty) \to C_{tem})$ of continuous, C_{tem}-valued processes by

$$d(f,g) = \sum_{T \in \mathbb{N}} 2^{-T} \sup_{s \leq T} d_{tem}(f(s), g(s)).$$

This will be useful later when we come to construct solutions by considering the convergence of approximating processes. We denote by C^+_{tem} the set of nonnegative functions in C_{tem}.

The concept of space-time white noise is discussed in Walsh [32] and we provide a brief review in the appendix. We follow Shiga [29] in defining solutions of (1.1) in the following Schwartz distribution sense. An $\{\mathcal{F}_t\}$-predictable function $u : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a *solution* of (1.1) with (deterministic) initial condition f if, for each $\phi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}(u(t),\phi) &= (f,\phi) + \int_0^t \left((u(s),\Delta\phi) + (a(u(s,.)),\phi)\right) ds \\ &+ \int_0^t \int b(u(s,x))\phi(x)\, W(dx\, ds)\end{aligned} \tag{1.2}$$

(where (f,g) is defined to be $\int f(x)g(x)\,dx$ for functions $f,g : \mathbb{R} \to \mathbb{R}$ where the integral makes sense). If in addition $u(t,\cdot) \in C_{tem}$ for each $t \geq 0$, and the map $t \mapsto u(t,\cdot)$ is continuous under the metric on C_{tem}, then u is known as a *C_{tem}-valued continuous solution.*

Under the condition (1.3) on a and b given below, Shiga [29] gives the following useful Green's function representation of u. Set

$$G(t,x,y) = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}$$

and let $G(t)f(x) = \int G(t,x,y)f(y)\,dy$ for suitable functions f. We also abbreviate $G(t,x,0)$ by $G(t,x)$.

Theorem 1.1.1 *(Theorem 2.1 of Shiga [29]) Suppose that $f \in C_{tem}$ and that there exists a constant L such that for all $u \in \mathbb{R}$,*

$$|a(u)| + |b(u)| \leq L(|u| + 1). \tag{1.3}$$

Then u is a C_{tem}-valued continuous solution of (1.1) with initial condition f if and only if $u(t,\cdot)$ is an $\{\mathcal{F}_t\}$-predictable and C_{tem}-valued continuous process that satisfies, for $t > 0$ and $x \in \mathbb{R}$, the following integral equation:

$$\begin{aligned} u(t,x) &= G(t)f(x) + \int_0^t \int G(t-s,x,y)a(u(s,y))\,dy\,ds \\ &+ \int_0^t \int G(t-s,x,y)b(u(s,y))\,W(dy\,ds). \end{aligned} \tag{1.4}$$

Theorem 2.2 of Shiga [29] is a more general version of the following existence and uniqueness theorem. A solution u of an SPDE is said to be *pathwise unique* if, given another solution v on the same space with respect to the same noise, $u = v$ almost surely.

Theorem 1.1.2 *Suppose a and b are Lipschitz. Then for every $f \in C_{tem}$, there exists a (pathwise) unique C_{tem}-valued continuous solution $u(t,x)$.*

It is often useful to have uniqueness in law, i.e. the probability distributions of two solutions, perhaps on different spaces, are the same. By following a procedure similar to that in the SODE case (see Theorem 17.1 on page 151 of Rogers and Williams [28]) it is possible to show that pathwise uniqueness implies uniqueness in law. This is done in Ondrejat [24]. In particular, uniqueness in law holds in the Lipschitz case. For equations with non-Lipschitz coefficients, it is still possible to show uniqueness in law in certain cases.

We now prove a lemma bounding moments of a solution in terms of the initial condition. A similar bound is buried in the proof of Theorem 2.2 in Shiga [29] but doesn't give an explicit dependence on the initial condition f.

Lemma 1.1.3 *Suppose a and b satisfy (1.3) and u is a C_{tem}-valued continuous solution of (1.1) with initial condition $f \in C_{tem}$. Then for all $T, \lambda > 0$ and $p \geq 2$, there exists a constant $C = C(T, \lambda, p, L)$ such that for all $z \in \mathbb{R}$,*

$$\sup_{s \leq T, y \in \mathbb{R}} \mathbb{E}\left[|u(s, y+z)|^p e^{-\lambda|y|}\right] \leq C \left(\sup_{y \in \mathbb{R}} |f(y+z)|^p e^{-\lambda|y|} + 1\right) < \infty.$$

Furthermore, if f is bounded, then

$$\sup_{s \leq T, y \in \mathbb{R}} \mathbb{E}[|u(s, y)|^p] \leq C \left(\sup_{y \in \mathbb{R}} |f(y)|^p + 1\right). \tag{1.5}$$

Proof Fix $p \geq 2$ and $\lambda > 0$. We define a sequence of stopping times τ_n by

$$\tau_n = \inf\left\{\, t > 0 \;:\; \sup_{y \in \mathbb{R}} |u(t, y)|^p e^{-\lambda|y|} \geq n \,\right\}.$$

Now τ_n is a well-defined stopping time for each $n \in \mathbb{N}$, and since $u(t)$ is continuous as a C_{tem}-valued process and $u(t)$ is defined for all $t \geq 0$, these times increase to infinity almost surely. Set

$$\begin{aligned} u^n(t,x) &= \int G(t,x,y)f(y)\,dy + \int_0^{t\wedge\tau_n}\int a(u(s,y))G(t-s,x,y)\,dy\,ds \\ &+ \int_0^{t\wedge\tau_n}\int b(u(s,y))G(t-s,x,y)W(dy\,ds). \end{aligned}$$

Then $u^n(t,x) = u(t,x)$ for $t \le \tau_n$. By Burkholder's inequality, for some $C = C(p)$,

$$\begin{aligned} \mathbb{E}[|u^n(t,x)|^p] &\le C\left(\int |f(y)|G(t,x,y)\,dy\right)^p \\ &+ C\mathbb{E}\left[\left(\int_0^{t\wedge\tau_n}\int |a(u(s,y))G(t-s,x,y)\,dy\,ds\right)^p\right] \\ &+ C\mathbb{E}\left[\left(\int_0^{t\wedge\tau_n}\int b(u(s,y))^2G(t-s,x,y)^2\,dy\,ds\right)^{p/2}\right]. \end{aligned}$$

Applying Hölder's inequality gives

$$\begin{aligned} &\mathbb{E}[|u^n(t,x)|^p] \\ &\le C\int |f(y)|^pG(t,x,y)\,dy \\ &+ C\mathbb{E}\left[\int_0^{t\wedge\tau_n}\int |a(u(s,y))|^pG(t-s,x,y)\,dy\,ds\right]\left(\int_0^t\int G(s,y)\,dy\,ds\right)^{p-1} \\ &+ C\mathbb{E}\left[\int_0^{t\wedge\tau_n}\int |b(u(s,y))|^pG(t-s,x,y)^2\,dy\,ds\right] \\ &\qquad\times\left(\int_0^t\int G(s,y)^2\,dy\,ds\right)^{p/2-1}. \end{aligned}$$

Since $\int G(s,y)\,dy = 1$ and $\int G(s,y)^2\,dy = \left(\sqrt{2}/4\sqrt{\pi s}\right)$, we have, for some $C = C(p,T)$,

$$\mathbb{E}[|u^n(t,x)|^p] \le C\int |f(y)|^pG(t,x,y)\,dy$$

$$+\quad C\mathbb{E}\left[\int_0^{t\wedge\tau_n}\int |a(u(s,y))|^p G(t-s,x,y)\,dy\,ds\right]$$
$$+\quad C\mathbb{E}\left[\int_0^{t\wedge\tau_n}\int |b(u(s,y))|^p G(t-s,x,y)^2\,dy\,ds\right].$$

For all $x, y \in \mathbb{R}$, $s > 0$ and $\lambda > 0$,

$$e^{-\lambda|x|}G(s,x,y) \leq e^{-\lambda|y|}e^{\lambda|y-x|}G(s,x,y). \tag{1.6}$$

For all $\alpha > 0$,

$$\frac{2|y-x|}{\lambda\alpha s} \leq \frac{|y-x|^2}{(\lambda\alpha s)^2} + 1$$

so

$$2\lambda|y-x| \leq \frac{|y-x|^2}{\alpha s} + \lambda^2\alpha s. \tag{1.7}$$

Choosing $\alpha = 8$, we have, by substituting (1.7) into (1.6),

$$\begin{aligned} e^{-\lambda|x|}G(s,x,y) &\leq \frac{e^{-\lambda|y|}e^{(y-x)^2/8s}e^{8s\lambda^2}e^{-(y-x)^2/4s}}{\sqrt{4\pi s}} \\ &= \left(\sqrt{2}\,e^{8s\lambda^2}\right)e^{-\lambda|y|}G(2s,x,y). \end{aligned} \tag{1.8}$$

Hence, for some C and C' depending only on p, λ, T,

$$\begin{aligned} &\mathbb{E}\left[|u^n(t,x)|^p e^{-\lambda|x|}\right] \\ &\leq C\int |f(y)|^p e^{-\lambda|x|}G(t,x,y)\,dy \\ &+ C\int_0^t\int \mathbb{E}\left[|a(u^n(s,y))|^p\chi_{\{s\leq\tau_n\}}\right]e^{-\lambda|x|}G(t-s,x,y)\,dy\,ds \\ &+ C\int_0^t\int \mathbb{E}\left[|b(u^n(s,y))|^p\chi_{\{s\leq\tau_n\}}\right]\left(e^{-\lambda|x|/2}G(t-s,x,y)\right)^2\,dy\,ds \\ &\leq C'\int |f(y)|^p e^{-\lambda|y|}G(2t,x,y)\,dy \\ &+ C'\int_0^t\int \mathbb{E}\left[|a(u^n(s,y))|^p\chi_{\{s\leq\tau_n\}}\right]e^{-\lambda|y|}G(2(t-s),x,y)\,dy\,ds \end{aligned}$$

$$
\begin{aligned}
&+ \quad C' \int_0^t \int \mathbb{E}\left[|b(u^n(s,y))|^p \chi_{\{s\le\tau_n\}}\right] e^{-\lambda|y|} G(2(t-s),x,y)^2\, dy\, ds \\
&\le \quad C' \left(\sup_{y\in\mathbb{R}} |f(y)|^p e^{-\lambda|y|}\right) \left(\int G(2t,y)\, dy\right) \\
&+ \quad C' \int_0^t \left(\int G(2(t-s),y)\, dy\right) \sup_{y\in\mathbb{R}} \mathbb{E}\left[|a(u^n(s,y))|^p e^{-\lambda|y|} \chi_{\{s\le\tau_n\}}\right] ds \\
&+ \quad C' \int_0^t \left(\int G(2(t-s),y)^2\, dy\right) \sup_{y\in\mathbb{R}} \mathbb{E}\left[|b(u^n(s,y))|^p e^{-\lambda|y|} \chi_{\{s\le\tau_n\}}\right] ds
\end{aligned}
$$

so for some $C = C(p,\lambda,T,L)$,

$$
\begin{aligned}
\sup_{y\in\mathbb{R}} \mathbb{E}\left[|u^n(t,y)|^p e^{-\lambda|y|}\right] &\le C\left(\sup_{y\in\mathbb{R}} |f(y)|^p e^{-\lambda|y|}\right) \\
&+ \quad C\int_0^t \frac{1}{\sqrt{t-s}} \sup_{y\in\mathbb{R}} \mathbb{E}\left[|u^n(s,y)|^p e^{-\lambda|y|}\chi_{\{s\le\tau_n\}} + 1\right] ds \qquad (1.9)
\end{aligned}
$$

and so, since $\sup_{y\in\mathbb{R}} |u^n(s,y)|^p e^{-\lambda|y|} \le n$ for $s \le \tau_n$,

$$
\sup_{y\in\mathbb{R}} \mathbb{E}\left[|u^n(s,y)|^p e^{-\lambda|y|}\right] \le C\left(\sup_{y\in\mathbb{R}} |f(y)|^p e^{-\lambda|y|} + \int_0^s \frac{n+1}{\sqrt{r}}\, dr\right) < \infty
$$

for all $s \le T$. We can now use Lemma 1.1.4, the generalisation of Gronwall's inequality given below, on (1.9) to bound

$$
\sup_{y\in\mathbb{R}} \mathbb{E}\left[|u^n(t,y)|^p e^{-\lambda|y|}\right]
$$

independently of n. Applying Fatou's Lemma completes the proof for $z = 0$. If we set $v(t,y) = u(t,y+z)$, then v is a solution with initial condition $f(\cdot + z)$ so the more general result follows. Now (1.5) follows by noting that for each $\lambda > 0$ and $z \in \mathbb{R}$,

$$
\sup_{s\le T} \mathbb{E}\left[|u(s,z)|^p\right] \le \sup_{s\le T, y\in\mathbb{R}} \mathbb{E}\left[|u(s,y+z)|^p e^{-\lambda|y|}\right]
$$

and applying the result above. □

Lemma 1.1.4 *Suppose $f : [0,T] \to \mathbb{R}$ is bounded and that there exist constants $C, \varepsilon \geq 0$ such that for all $t \in [0,T]$,*

$$0 \leq f(t) \leq \varepsilon + C \int_0^t \frac{f(s)}{\sqrt{t-s}}\, ds. \tag{1.10}$$

Then $f(t) \leq (1 + 2Ct^{1/2})\varepsilon e^{C^2 \pi t}$ for all $t \in [0,T]$.

Proof This is a generalisation of Gronwall's inequality where we apply the following trick from page 314 of Walsh [32] to deal with the $(t-s)^{-1/2}$ part (see also Lemma 7.1.1 of Henry [10]). Iterating (1.10) gives

$$\begin{aligned}
f(t) &\leq (1 + 2Ct^{1/2})\varepsilon + C^2 \int_0^t \int_0^s \frac{f(r)}{\sqrt{(t-s)(s-r)}}\, dr\, ds \\
&= (1 + 2Ct^{1/2})\varepsilon + C^2 \int_0^t \left(\int_r^t \frac{1}{\sqrt{(t-s)(s-r)}}\, ds \right) f(r)\, dr \\
&= (1 + 2Ct^{1/2})\varepsilon + C^2 \int_0^t \left(\int_0^1 \frac{1}{\sqrt{(1-s)s}}\, ds \right) f(r)\, dr \\
&= (1 + 2Ct^{1/2})\varepsilon + C^2 \pi \int_0^t f(r)\, dr.
\end{aligned}$$

The result follows by applying Gronwall's inequality. □

We now show that for a certain class of solutions, we can generalise the formula (1.2) to allow us to integrate against more general functions. Set

$$G(t, \phi, y) = \int G(t, x, y) \phi(x)\, dx \tag{1.11}$$

for functions $\phi : \mathbb{R} \to \mathbb{R}$ where the integral makes sense.

Definition 1.1.5 *Let $\psi = \psi(t, x)$ be a function in $C^{1,2}([0,T] \times \mathbb{R})$ for some $T > 0$. We say that ψ is a* test function *if $\psi(s, \cdot) \in C^2(\mathbb{R})$, the following conditions are* *satisfied for some* $\lambda > 0$*:*

$$\int \left(|\psi(t,x)| + \psi(t,x)^2\right) e^{\lambda|x|}\, dx < \infty \text{ for each } t \in [0,T], \tag{1.12}$$

$$\int_0^T \int \left(|\Delta\psi + \partial_t\psi| + |\Delta\psi + \partial_t\psi|^2\right)(s,x) e^{\lambda|x|}\, dx\, ds < \infty, \tag{1.13}$$

and for all $y \in \mathbb{R}$ *and* $0 \le s < t$,

$$G(t-s, \psi(t,\cdot), y) - \psi(s,y) - \int_s^t G(r-s, (\Delta\psi + \partial_t\psi)(r,\cdot), y)\, dr = 0. \tag{1.14}$$

By integration by parts and using the fact that $\partial_t G = \Delta G$, it is possible to prove (1.14) if we make further assumptions on the behaviour of $\psi(t,x)$ and its derivatives as $x \to \pm\infty$. However, it will be more convenient to check (1.14) directly for individual ψ as needed.

Lemma 1.1.6 *Suppose* u *is a* C_{tem}*-valued continuous solution of (1.1) with* a *and* b *satisfying (1.3), and initial condition* $f \in C_{tem}$*. If* ψ *is a test function then for each* $t \in [0,T]$ *we have*

$$\begin{aligned}(u(t),\psi(t)) &= (f,\psi(0)) + \int_0^t (u(s), (\Delta + \partial_t)\psi(s))\, ds + \\ &\quad \int_0^t (a(u(s,.)), \psi(s))\, ds + \int_0^t \int b(u(s,y))\psi(s,y)\, W(dy\, ds).\end{aligned} \tag{1.15}$$

Proof We follow a method similar to that in Walsh [32], on page 317 in Theorem 3.2. Plugging the Green's function formula (1.4) into the left hand side minus the right hand side of equation (1.15) gives

$$\int \psi(t,x) \left[\int G(t,x,y) f(y)\, dy + \int_0^t \int G(t-s,x,y) a(u(s,y))\, dy\, ds + \right.$$
$$\left. \int_0^t \int G(t-s,x,y) b(u(s,y))\, W(dy\, ds)\right] dx -$$

$$
\begin{aligned}
&\int f(x)\psi(0,x)\,dx - \int_0^t \int a(u(s,x))\psi(s,x)\,dx\,ds - \\
&\qquad \int_0^t \int b(u(s,x))\psi(s,x)\,W(dx\,ds) - \\
&\int_0^t \int (\Delta\psi + \partial_t\psi)(s,x) \left[\int G(s,x,y)f(y)\,dy + \right. \\
&\qquad \int_0^s \int G(s-r,x,y)a(u(r,y))\,dy\,dr + \\
&\qquad \left. \int_0^s \int G(s-r,x,y)b(u(r,y))\,W(dy\,dr) \right] dx\,ds.
\end{aligned} \tag{1.16}
$$

We wish to rearrange the multiple integrals to integrate over x first. To do this, we need to check the hypotheses of Fubini's theorem and its stochastic analogue (Theorem 2.6 of Walsh [32]). We only treat the stochastic integral terms here, since the others are straightforward. Using (1.8), for some $C = C(T, \lambda)$,

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t \int \int G(t-s,x,y)^2 b(u(s,y))^2 \psi(t,x)^2\,dy\,dx\,ds \right] \\
&\quad \le \; L^2 \int_0^t \int \int G(t-s,x,y)^2 \mathbb{E}\left[(|u(s,y)|+1)^2 \right] e^{-\lambda|x|} \psi(t,x)^2 e^{\lambda|x|}\,dy\,dx\,ds \\
&\quad \le \; CL^2 \sup_{s\le T, y\in\mathbb{R}} \mathbb{E}\left[(|u(s,y)|+1)^2 e^{-\lambda|y|} \right] \int \psi(t,x)^2 e^{\lambda|x|}\,dx
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t \int_0^s \int \int (\Delta\psi + \partial_t\psi)^2 (s,x) G(s-r,x,y)^2 b(u(r,y))^2\,dy\,dx\,dr\,ds \right] \\
&\quad \le \; CL^2 \sup_{s\le T, y\in\mathbb{R}} \mathbb{E}\left[(|u(s,y)|+1)^2 e^{-\lambda|y|} \right] \int_0^t \int (\Delta\psi + \partial_t\psi)^2 (s,x) e^{\lambda|x|}\,dx\,ds
\end{aligned}
$$

which are both finite by the definition of a test function and Lemma 1.1.3. Hence (1.16) is equal to

$$
\int G(t,\psi(t),y)f(y)\,dy + \int_0^t \int G(t-s,\psi(t),y)a(u(s,y))\,dy\,ds +
$$

$$\int_0^t \int G(t-s, \psi(t), y) b(u(s,y))\, W(dy\,ds) -$$
$$\int_0^t \int G(s, (\Delta\psi + \partial_t\psi)(s), y) f(y)\, dy\, ds -$$
$$\int_0^t \int_0^s \int G(s-r, (\Delta\psi + \partial_t\psi)(s), y) a(u(r,y))\, dy\, dr\, ds -$$
$$\int_0^t \int_0^s \int G(s-r, (\Delta\psi + \partial_t\psi)(s), y) b(u(r,y))\, W(dy\,dr)\, ds -$$
$$\int f(y)\psi(0,y)\, dy - \int_0^t \int a(u(s,y))\psi(s,y)\, dy\, ds -$$
$$\int_0^t \int b(u(s,y))\psi(s,y)\, W(dy\,ds).$$

Applying Fubini's theorem again, we see that this is equal to

$$\int f(y) \left[G(t, \psi(t), y) - \psi(0,y) - \int_0^t G(s, (\Delta\psi + \partial_t\psi)(s), y)\, ds \right] dy +$$
$$\int_0^t \int a(u(s,y)) \left[G(t-s, \psi(t), y) - \psi(s,y) - \right.$$
$$\left. \int_s^t G(r-s, (\Delta\psi + \partial_t\psi)(r), y)\, dr \right] dy\, ds +$$
$$\int_0^t \int b(u(s,y)) \left[G(t-s, \psi(t), y) - \psi(s,y) - \right.$$
$$\left. \int_s^t G(r-s, (\Delta\psi + \partial_t\psi)(r), y)\, dr \right] W(dy\,ds).$$

Each of the three terms in square brackets now vanishes by (1.14). □

It will be useful later to have the following alternate Green's function representation for the case where $a(u) = \tilde{a}(u) - \theta u$ for some easily controllable $\tilde{a}$ (for example, $\tilde{a}$ bounded). Let $G^\theta(t,x,y) = e^{-\theta t} G(t,x,y)$.

Lemma 1.1.7 *Suppose u is a C_{tem}-valued continuous solution to (1.1) with a, b satisfying (1.3) and $u(0) = f$. Set $\tilde{a}(u) = a(u) + \theta u$ for some constant $\theta \in \mathbb{R}$*

Then, for each $x \in \mathbb{R}$, $t > 0$,

$$u(t,x) = \int G^\theta(t,x,y)f(y)\,dy + \int_0^t \int G^\theta(t-s,x,y)\bar{a}(u(s,y))\,dy\,ds$$
$$+ \int_0^t \int G^\theta(t-s,x,y)b(u(s,y))\,W(dy\,ds). \tag{1.17}$$

Proof For $n \in \mathbb{N}$, choose $\xi_n(s,y) = G^\theta(s+1/n,x,y)$. Then, for each $y \in \mathbb{R}$ and $0 \le s < t$,

$$\xi_n(t,y) = \int G(t-s,y,z)\xi_n(s,z)\,dz - \tag{1.18}$$
$$\theta \int_0^{t-s} \int G(t-s-r,y,z)\xi_n(s+r,z)\,dz\,dr.$$

For a fixed $T > 0$, set $\psi_n(s,y) = \xi_n(T-s,y) = G^\theta(T-s+1/n,x,y)$. Then $\partial_t\psi_n + \Delta\psi_n - \theta\psi_n = 0$, and ψ_n is a test function on $[0,T]$ since (1.14) follows by reversing (1.18), so plugging into (1.15) gives

$$\int u(t,y)G^\theta(1/n,x,y)\,dy = \int f(y)G^\theta(t+1/n,x,y)\,dy$$
$$+ \int_0^t \int \bar{a}(u(s,\cdot))G^\theta(t-s+1/n,x,y)\,dy\,ds$$
$$+ \int_0^t \int b(u(s,y))G^\theta(t-s+1/n,x,y)\,W(dy\,ds).$$

Taking $n \to \infty$, we check that the left-hand side and the first two terms on the right each converge almost surely, and the third term on the right converges in L^2. Then each side converges in probability and the result follows. □

1.2 A comparison result

It is often useful, and it will be important to us later on, to be able to tell when, given two solutions u and v of an SPDE with $u(0) \geq v(0)$, $u(t)$ stays above $v(t)$ for all t. Such comparison results can be found in Donati-Martin and Pardoux [5], Kotelenez [13], Mueller [15] and Shiga [29]. We generalise and consider the system

$$\left.\begin{aligned} \partial_t u &= \Delta u + a_1(u) + b(u)\, dW_1, \\ \partial_t v &= \Delta v + a_2(v) + b(v)\left[\alpha(u-v)\, dW_1 + \beta(u-v)\, dW_2\right] \end{aligned}\right\} \tag{1.19}$$

for independent white noises W_1 and W_2, where $a_1 \geq a_2$, $\alpha^2 + \beta^2 = 1$ and $\beta(0) = 0$. Note here that $\alpha, \beta : \mathbb{R} \to [0,1]$ are functions. For the time being, we assume all the coefficients a_1, a_2, b, α and β are Lipschitz, although we will consider more general coefficients later. We also assume that there exists a constant C such that for all $u, u', v, v' \in \mathbb{R}$ we have

$$\left.\begin{aligned} |b(v)\alpha(u-v) - b(v')\alpha(u'-v')| &\leq C(|u-u'| + |v-v'|), \\ |b(v)\beta(u-v) - b(v')\beta(u'-v')| &\leq C(|u-u'| + |v-v'|). \end{aligned}\right\} \tag{1.20}$$

In particular, (1.20) holds if either b is bounded or $\alpha \equiv 1$, which happen to be the two cases we will be interested in later.

u and v are considered to be solutions in an analogous way to the single noise case, and the existence of a pair (u, v) which solves this system can be established in a similar way. We consider two noises to allow us to prove a coupling result later. Let

$$W = \alpha(u-v)W_1 + \beta(u-v)W_2.$$

Then v is a solution of (1.1) with respect to W. Since W_1 and W_2 are independent white noises, it is easy to see that if $A \cap B = \emptyset$ then $W(A)_t$ and $W(B)_t$ are independent and $W(A \cup B)_t = W(A)_t + W(B)_t$. Also,

$$\begin{aligned}
\langle W(A) \rangle_t &= \left\langle \int_0^\cdot \int_A \alpha((u-v)(s,x))\, W_1(dx\, ds) \right\rangle_t \\
&+ \left\langle \int_0^\cdot \int_A \beta((u-v)(s,x))\, W_2(dx\, ds) \right\rangle_t \\
&= \int_0^t \int_A \left(\alpha^2((u-v)(s,x)) + \beta^2((u-v)(s,x)) \right) dx\, ds
\end{aligned}$$

which is equal to $t|A|$ since $\alpha^2 + \beta^2 = 1$. Hence W is also a standard white noise.

We give some definitions and a lemma from Shiga [29] prior to stating the result. For each $\varepsilon > 0$, we set G_ε to be the operator

$$G_\varepsilon(t) = e^{-t/\varepsilon} I + R_\varepsilon(t)$$

where $R_\varepsilon(t)$ is an integral operator with kernel

$$R_\varepsilon(t,x,y) = e^{-t/\varepsilon} \sum_{n=1}^{\infty} \frac{(t/\varepsilon)^n}{n!} G(n\varepsilon, x, y).$$

We see because $\int G(t,x,y)\, dy = 1$ that $\int R_\varepsilon(t,y)\, dy = 1 - e^{-t/\varepsilon} < 1$. Also, if $N_{\varepsilon,t}$ is a Poisson(t/ε)-distributed random variable and B is an independent Brownian motion, we have, for continuous functions f,

$$G_\varepsilon(t) f(x) = \mathbb{E}\left[G(\varepsilon N_{\varepsilon,t}) f(x) \right] = \mathbb{E}\left[f(x + B(\varepsilon N_{\varepsilon,t})) \right].$$

Lemma 1.2.1 *(Lemma 6.6 of Shiga [29])*

1. *For all* $t > 0$, $x \in \mathbb{R}$ *and* $\varepsilon \in (0,1]$,

$$\int R_\varepsilon(t,x,y)^2\, dy \le \sqrt{\frac{3}{8\pi}}\, t^{-1/2}.$$

2. *There exists $\lambda > 0$ and $\mu > 0$ such that for $0 < \frac{\varepsilon}{t} \leq \mu$,*

$$\int |R_\varepsilon(t,x,y) - G(t,x,y)|\,dy \leq e^{-t/\varepsilon} + \lambda \left(\frac{\varepsilon}{t}\right)^{1/3}.$$

3. *For $t > 0$ and $x \in \mathbb{R}$,*

$$\lim_{\varepsilon \to 0} \int_0^t \int (R_\varepsilon(s,x,y) - G(s,x,y))^2\,dy\,ds = 0.$$

We now state and prove our comparison result.

Proposition 1.2.2 *Suppose (u, v) is a pair of continuous, C_{tem}-valued functions which solves the above system, and $(u(0), v(0)) = (f, g)$ where $f, g \in C_{tem}$ and $f \geq g$. Suppose that the coefficients satisfy the conditions given above. Then, with probability one, $u(t) \geq v(t)$ for all $t \geq 0$.*

Proof The proof follows the ideas of Theorem 2.3 of Shiga [29], with a generalisation to cover the extra noise W_2. The idea of Shiga's method is to apply Ito's formula to the process $(u - v)(t, x)$ for fixed x to show that it is almost surely nonnegative. However, this process is too rough to be a semimartingale, so we first prove the result for a system with the noises smoothed by a parameter ε and show that we can approximate our original system by sending ε to zero.

We first assume f and g are bounded and uniformly continuous. Set $U = u - v$. For each $\varepsilon > 0$ choose a smooth, symmetric function $\rho_\varepsilon : \mathbb{R} \to \mathbb{R}_+$ such that $\int \rho_\varepsilon = 1$ and $\rho_\varepsilon \leq \varepsilon^{-1/2}\chi_{(-\varepsilon^{-1/2}, \varepsilon^{1/2})}$. This gives us $\int \rho_\varepsilon^2 \leq 2\varepsilon^{-1/2}$. For each $x \in \mathbb{R}$ and $i = 1$ or 2, we define a smoothed noise process

$$W_{i,x}^\varepsilon(t) = \int_0^t \int \rho_\varepsilon(x - z)\,W_i(dz\,ds).$$

Note that $\langle W^{\varepsilon}_{i,x} \rangle_t = \left(\int \rho_\varepsilon^2\right) t$, so for each x, $W^{\varepsilon}_{i,x}(t)$ is a Brownian motion sped up by a factor of $\int \rho_\varepsilon^2$. We also define a smoothed Laplacian operator Δ_ε to be

$$\Delta_\varepsilon = \frac{1}{\varepsilon}(G(\varepsilon) - I).$$

Note that for all $t > 0$, $y \in \mathbb{R}$ and a suitably large class of functions ϕ we have

$$\frac{\partial}{\partial t} G_\varepsilon(t, \phi, y) = G_\varepsilon(t, \Delta_\varepsilon \phi, y). \tag{1.21}$$

where $G_\varepsilon(t, \phi, y) := G_\varepsilon(t)\phi(y)$. By a standard iterative approximation procedure similar to what can be found in, say, Shiga [29] or Walsh [32], we can construct a solution $(u_\varepsilon, v_\varepsilon)$ to the system

$$\begin{aligned}
u_\varepsilon(t,x) &= G_\varepsilon(t) f(x) \\
&+ \int_0^t e^{-(t-s)/\varepsilon} a_1(u_\varepsilon(s,x))\, ds \\
&+ \int_0^t \int R_\varepsilon(t-s,x,y) a_1(u_\varepsilon(s,y))\, dy\, ds \\
&+ \int_0^t e^{-(t-s)/\varepsilon} b(u_\varepsilon(s,x))\, dW^{\varepsilon}_{1,x}(s) \\
&+ \int \int_0^t R_\varepsilon(t-s,x,y) b(u_\varepsilon(s,y))\, dW^{\varepsilon}_{1,y}(s)\, dy. \\
v_\varepsilon(t,x) &= G_\varepsilon(t) g(x) \\
&+ \int_0^t e^{-(t-s)/\varepsilon} a_2(v_\varepsilon(s,x))\, ds \\
&+ \int_0^t \int R_\varepsilon(t-s,x,y) a_2(v_\varepsilon(s,y))\, dy\, ds \\
&+ \int_0^t e^{-(t-s)/\varepsilon} b(v_\varepsilon(s,x)) \big[\alpha(U_\varepsilon(s,x))\, dW^{\varepsilon}_{1,x}(s) + \\
&\qquad \beta(U_\varepsilon(s,x))\, dW^{\varepsilon}_{2,x}(s)\big] \\
&+ \int \int_0^t R_\varepsilon(t-s,x,y) b(v_\varepsilon(s,y)) \big[\alpha(U_\varepsilon(s,y))\, dW^{\varepsilon}_{1,y}(s) + \\
&\qquad \beta(U_\varepsilon(s,y))\, dW^{\varepsilon}_{2,y}(s)\big]\, dy
\end{aligned}$$

such that for any $T > 0$ and $\varepsilon > 0$,

$$\sup_{t \le T, x \in \mathbb{R}} \mathbb{E}\left[u_\varepsilon(s,x)^2 + v_\varepsilon(s,x)^2\right] < \infty.$$

We would like this moment bound to be independent of ε. It follows from the Green's function representation given above that there exists a constant C depending on T, a, b and α such that for each $\varepsilon > 0$ and $t \le T$, $\mathbb{E}[u_\varepsilon(t,x)^2 + v_\varepsilon(t,x)^2]$ is bounded by C times

$$\begin{aligned}
&(G_\varepsilon(t)f(x))^2 + (G_\varepsilon(t)g(x))^2 \\
&\quad + \left(\int \rho_\varepsilon^2\right) \int_0^t e^{-2(t-s)/\varepsilon} \mathbb{E}\left[u_\varepsilon(s,x)^2 + v_\varepsilon(s,x)^2 + 1\right] ds \\
&\quad + \int_0^t R_\varepsilon(t-s,x,y) \mathbb{E}\left[u_\varepsilon(s,y) + v_\varepsilon(s,y)^2 + 1\right] dy\, ds \\
&\quad + \int_0^t \int R_\varepsilon(t-s,x,y)^2 \mathbb{E}\left[u_\varepsilon(s,y)^2 + v_\varepsilon(s,y)^2 + 1\right] dy\, ds.
\end{aligned}$$

Since $e^{-2s/\varepsilon} \int \rho_\varepsilon^2 \le 2e^{-2s/\varepsilon}\varepsilon^{-1/2}$, which we can bound by $(es)^{-1/2}$ for all $\varepsilon > 0$ and $s > 0$ by a little calculus, we see that for some (possibly different) $C = C(T, a, b, \alpha)$,

$$\begin{aligned}
&\sup_{y \in \mathbb{R}} \mathbb{E}\left[u_\varepsilon(t,y)^2 + v_\varepsilon(t,y)^2 + 1\right] \\
&\qquad \le C\left(\sup f^2 + \sup g^2\right) \\
&\qquad + C \int_0^t \frac{1}{\sqrt{t-s}} \sup_{y \in \mathbb{R}} \mathbb{E}\left[u_\varepsilon(s,y)^2 + v_\varepsilon(s,y)^2 + 1\right] ds.
\end{aligned}$$

Applying Gronwall's inequality, we see that

$$\sup_{t \le T, x \in \mathbb{R}, \varepsilon \in (0,1]} \mathbb{E}\left[u_\varepsilon(s,x)^2 + v_\varepsilon(s,x)^2\right] < \infty.$$

By a similar method as for Lemma 1.1.6 earlier, and using (1.21) we can show that for a suitable class of functions we have

$$\begin{aligned}
(u(t),\phi) &= (u_0,\phi) + \int_0^t \int (\Delta_\varepsilon u_\varepsilon(s,x) + a_1(u_\varepsilon(s,x)))\phi(x)\,dx\,ds \\
&+ \int\int_0^t b(u_\varepsilon(s,x))\phi(x)\,dW^\varepsilon_{1,x}(s)\,dx, \\
(v(t),\phi) &= (v_0,\phi) + \int_0^t \int (\Delta_\varepsilon v_\varepsilon(s,x) + a_2(v_\varepsilon(s,x)))\phi(x)\,dx\,ds \\
&+ \int\int_0^t b(v_\varepsilon(s,x))\phi(x)\left[\alpha(U_\varepsilon(s,x))\,dW^\varepsilon_{1,x}(s) + \right. \\
&\qquad \left. \beta(U_\varepsilon(s,x))\,dW^\varepsilon_{2,x}(s)\right]\,dx.
\end{aligned}$$

By taking a suitable sequence ϕ_n approximating δ_x, for example

$$\phi_n = \frac{n}{2}\chi_{x+(-\frac{1}{n},\frac{1}{n})},$$

we get the following integral equations.

$$\begin{aligned}
u_\varepsilon(t,x) = f(x) &+ \int_0^t (\Delta_\varepsilon u_\varepsilon(s,x) + a_1(u_\varepsilon(s,x)))\,ds \\
&+ \int_0^t b(u_\varepsilon(s,x))\,dW^\varepsilon_{1,x}(s) \\
v_\varepsilon(t,x) = g(x) &+ \int_0^t (\Delta_\varepsilon v_\varepsilon(s,x) + a_2(v_\varepsilon(s,x)))\,ds \\
&+ \int_0^t b(v_\varepsilon(s,x))\left[\alpha(U_\varepsilon(s,x))\,dW^\varepsilon_{1,x}(s) + \right. \\
&\qquad \left. \beta(U_\varepsilon(s,x))\,dW^\varepsilon_{2,x}(s)\right].
\end{aligned}$$

So, setting $U_\varepsilon = u_\varepsilon - v_\varepsilon$,

$$\begin{aligned}
dU_\varepsilon(t,x) &= (\Delta_\varepsilon U_\varepsilon(t,x) + a_1(u_\varepsilon(t,x)) - a_2(v_\varepsilon(t,x)))\,dt \\
&+ (b(u_\varepsilon(t,x)) - b(v_\varepsilon(t,x))\alpha(U_\varepsilon(t,x)))\ dW^\varepsilon_{1,x}(t) \\
&- b(v_\varepsilon(t,x))\beta(U_\varepsilon(t,x))\,dW^\varepsilon_{2,x}(t)
\end{aligned}$$

and

$$\begin{aligned} d\langle U_\varepsilon(\cdot,x)\rangle_t &= \left(\int \rho_\varepsilon^2\right)\left[(b(u_\varepsilon(t,x)) - b(v_\varepsilon(t,x)))^2\right. \\ &+ \left. 2b(u_\varepsilon(t,x))b(v_\varepsilon(t,x))(1-\alpha(U_\varepsilon(t,x)))\right]\,dt. \end{aligned}$$

Since α is Lipschitz and $1-\alpha(U) = \alpha(0)-\alpha(U)$, we have

$$\begin{aligned} d\langle U_\varepsilon(\cdot,x)\rangle_t &\leq C\,(|b(u_\varepsilon(t,x)) - b(v_\varepsilon(t,x))||U_\varepsilon(t,x)| \\ &\qquad +2|b(u_\varepsilon)b(v_\varepsilon(t,x))||U_\varepsilon(t,x)|)\;dt \\ &\leq C'(|u_\varepsilon(t,x)| + |v_\varepsilon(t,x)| + 1)^2|U_\varepsilon(t,x)|\,dt \end{aligned}$$

for some constants C, C' depending on α, b and ε.

The next stage is to show that $u_\varepsilon \geq v_\varepsilon$ almost surely. Set $V_\varepsilon = v_\varepsilon - u_\varepsilon = -U_\varepsilon$. We show that $\mathbb{E}[V_\varepsilon(t,x)^+] = 0$ for each $t \geq 0$ and $x \in \mathbb{R}$ by using the procedure in Ikeda and Watanabe [12] to approximate the function $x \mapsto x^+$ by a sequence of C^2 functions as follows. Since $\int_0^1 x^{-1}\,dx = \infty$, we can choose a decreasing sequence of real numbers (k_n) such that for each $n \in \mathbb{N}$, $\int_{k_{n+1}}^{k_n} x^{-1}\,dx \geq 2n$. Then we can choose a sequence of functions $\phi_n \in C^+(\mathbb{R})$ such that

- $\int \phi_n = 1$
- $\phi_n(x) \leq \frac{1}{nx}$ for $x > 0$
- supp $\phi_n \subseteq [k_{n+1}, k_n]$

We now let

$$\psi_n(x) = \int_0^x \int_0^y \phi_n(z)\,dz\,dy.$$

Then $\psi_n(x) \nearrow x^+$, $\psi_n' \nearrow \chi_{(0,\infty)}$ and $\psi_n'' = \phi_n$. Applying Ito's formula gives, for some constant C, which is independent of n and t up to a fixed time T and may change from line to line, and some local martingale M,

$$\begin{aligned}
\psi_n(V_\varepsilon(t,x)) &\leq \int_0^t \psi_n'(V_\varepsilon(s,x))\,(\Delta_\varepsilon V_\varepsilon(s,x) + a_2(v_\varepsilon(s,x)) - a_1(u_\varepsilon(s,x)))\,ds \\
&+ C\int_0^t \phi_n(V_\varepsilon(s,x))|V_\varepsilon(s,x)|(|v_\varepsilon(s,x)|^2 + |u_\varepsilon(s,x)|^2 + 1)\,ds \\
&+ M(t).
\end{aligned}$$

The local martingale is easily seen to be a true martingale by the moment bound in Lemma 1.1.3. Note that $a_2(v_\varepsilon) - a_1(u_\varepsilon) \leq a_2(v_\varepsilon) - a_2(u_\varepsilon) \leq |V_\varepsilon|$. Also, $|V|\psi_n'(V)$ is bounded by V^+ and $|V|\phi_n(V)$ is bounded by $1/n$. Taking expectations, we get

$$\begin{aligned}
\mathbb{E}[\psi_n(t,x)] &\leq C\mathbb{E}\left[\int_0^t V_\varepsilon(s,x)^+\,ds + \int_0^t \psi_n'(V_\varepsilon(s,x))\Delta_\varepsilon V_\varepsilon(s,x)\,ds\right] \\
&+ \frac{Ct}{n}\sup_{s\leq t}\mathbb{E}\left[|u_\varepsilon(s,x)|^2 + |v_\varepsilon(s,x)|^2 + 1\right].
\end{aligned}$$

Since $|\Delta_\varepsilon f(x)| \leq \varepsilon^{-1}\sup_{y\in\mathbb{R}}|f(y)|$,

$$\mathbb{E}[\psi_n(t,x)] \leq C\left(\int_0^t \sup_{y\in\mathbb{R}}\mathbb{E}\left[V_\varepsilon(s,y)^+\right]\,ds + \frac{t}{n}\right).$$

Applying Fatou's Lemma, we get

$$\sup_{y\in\mathbb{R}}\mathbb{E}\left[V_\varepsilon(s,y)^+\right] \leq C\int_0^t \sup_{y\in\mathbb{R}}\mathbb{E}\left[V_\varepsilon(s,y)^+\right]\,ds$$

so $u_\varepsilon \geq v_\varepsilon$ almost surely by Gronwall's inequality.

The next stage in the proof is to show that our smoothed processes u_ε and v_ε approximate u and v, in the sense that for each fixed time $T > 0$,

$$\lim_{\varepsilon\to 0}\sup_{t\leq T, x\in\mathbb{R}}\mathbb{E}\left[|u(t,x) - u_\varepsilon(t,x)|^2 + |v(t,x) - v_\varepsilon(t,x)|^2\right] = 0. \tag{1.22}$$

The comparison result for u and v will then immediately follow. Using the Green's function equations, we see that $v_\varepsilon(t,x) - v(t,x)$ is equal to

$$
\begin{aligned}
&(G_\varepsilon(t)g(x) - G(t)g(x)) + \\
&\int_0^t e^{-(t-s)/\varepsilon} a_2(v_\varepsilon(s,x))\, ds + \\
&\int_0^t \int R_\varepsilon(t-s,x,y)(a_2(v_\varepsilon(s,y)) - a_2(v(s,y)))\, dy\, ds + \\
&\int_0^t \int (R_\varepsilon(t-s,x,y) - G(t-s,x,y)) a_2(v(s,y))\, dy\, ds + \\
&\int_0^t \int e^{-(t-s)/\varepsilon} b(v_\varepsilon(s,x))\alpha(U_\varepsilon(s,x))\rho_\varepsilon(y-x)\, W_1(dy\, ds) + \\
&\int_0^t \int \Big(\int R_\varepsilon(t-s,x,z)(b(v_\varepsilon(s,z))\alpha(U_\varepsilon(s,z)) - \\
&\qquad\qquad b(v(s,z))\alpha(U(s,z)))\rho_\varepsilon(y-z)\, dz \Big)\, W_1(dy\, ds) + \\
&\int_0^t \int \Big(\int R_\varepsilon(t-s,x,z)(b(v(s,z))\alpha(U(s,z)) - \\
&\qquad\qquad b(v(s,y))\alpha(U(s,y)))\rho_\varepsilon(y-z)\, dz \Big)\, W_1(dy\, ds) + \\
&\int_0^t \int \Big(\int R_\varepsilon(t-s,x,z)\rho_\varepsilon(y-z)\, dz - G(t-s,x,y) \Big) \\
&\qquad\qquad b(v(s,y))\alpha(U(s,y))\, W_1(dy\, ds) + \\
&J_9 + J_{10} + J_{11} + J_{12}.
\end{aligned}
$$

Let the first eight terms be $J_1(\varepsilon,t,x), \ldots, J_8(\varepsilon,t,x)$ respectively. Then $J_9 + \cdots + J_{12}$ is the same as $J_5 + \cdots + J_8$ with α and W_1 replaced by β and W_2. We show that for each $i = 1, \ldots, 8$, $\mathbb{E} J_i^2(\varepsilon,t,x)$ converges to zero uniformly in $t \leq T$ and $x \in \mathbb{R}$, except for the cases $i = 3$ and $i = 6$, which we control by a Gronwall argument. In the following, $C = C(a, b. \alpha, T)$ is a constant which may vary from line to line.

Let $N(\varepsilon, t)$ be a Poisson(t/ε) random variable and let B is an independent Brownian motion. Given $\gamma > 0$, choose $\delta > 0$ so that $|g(x) - g(y)| < \gamma$ for $|x - y| \leq \delta$. Then

$$\begin{aligned} |G_\varepsilon(t)g(x) - G(t)g(x)| &\leq \mathbb{E}[|g(x + B(\varepsilon N(\varepsilon, t))) - g(x + B(t))|] \\ &\leq \gamma + (2 \sup |g|)\mathbb{P}(|B(\varepsilon N(\varepsilon, t)) - B(t)| \geq \delta). \end{aligned}$$

Now

$$\mathbb{E}\left[|B(\varepsilon N(\varepsilon, t)) - B(t)|^2\right] = \mathbb{E}[|\varepsilon N(\varepsilon, t) - t|] \leq \varepsilon \mathrm{Var}(N(t/\varepsilon))^{1/2} = (\varepsilon t)^{1/2}$$

so $\lim_{\varepsilon \to 0} \sup_{t \leq T} \mathbb{P}(|B(\varepsilon N(\varepsilon, t)) - B(t)| \geq \delta) = 0$ for each $\delta > 0$. Therefore $|J_1(\varepsilon, t, x)|$ converges to zero as $\varepsilon \to 0$, uniformly for $(t, x) \in [0, T] \times \mathbb{R}$. For all $\varepsilon \in (0, 1]$, $x \in \mathbb{R}$ and $t \leq T$, we have

$$\begin{aligned} \mathbb{E}\left[J_2^2(\varepsilon, t, x)\right] &\leq \sup_{y \in \mathbb{R}, s \leq T} \mathbb{E}\left[a_2^2(v_\varepsilon(s, y))\right] \int_0^t e^{-s/\varepsilon}\, ds, \\ \mathbb{E}\left[J_5^2(\varepsilon, t, x)\right] &\leq \sup_{y \in \mathbb{R}, s \leq T} \mathbb{E}\left[b^2(v_\varepsilon(s, y))\right] \left(\int \rho_\varepsilon^2\right) \int_0^t e^{-2s/\varepsilon}\, ds \\ &\leq \sup_{y \in \mathbb{R}, s \leq T} \mathbb{E}\left[b^2(v_\varepsilon(s, y))\right] \sqrt{\varepsilon} \end{aligned}$$

which converge to zero as $\varepsilon \to 0$. Now

$$\mathbb{E} J_4^2(\varepsilon, t, x) \leq C \sup_{s \leq T, y \in \mathbb{R}} \mathbb{E}\left[|v(s, y)|^2 + 1\right] \left(\int_0^t \int |R_\varepsilon(s, y) - G(s, y)|\, dy\, ds\right).$$

By splitting the time integral into the regions $s \in [0, \varepsilon/\mu]$ and $s \in (\varepsilon/\mu, t]$ and applying part 2 of Lemma 1.2.1 we see that this converges to zero as $\varepsilon \to 0$.

$\mathbb{E}\left[J_7^2(\varepsilon, t, x)\right]$

$$\leq C \int_0^t \int \mathbb{E}\left[\left(\int R_\varepsilon(t-s,x,z)|u(s,z)-u(s,y)|\rho_\varepsilon(y-z)\,dz\right)^2\right] dy\,ds$$
$$+ C \int_0^t \int \mathbb{E}\left[\left(\int R_\varepsilon(t-s,x,z)|v(s,z)-v(s,y)|\rho_\varepsilon(y-z)\,dz\right)^2\right] dy\,ds.$$

We only deal with the u term, since the other is entirely similar. Splitting $R_\varepsilon = (R_\varepsilon - G) + G$ and applying Jensen's inequality, we see that this is bounded by C times

$$\int_0^t \int \int G(t-s,x,z)^2 \mathbb{E}\left[|u(s,z)-u(s,y)|^2\right] \rho_\varepsilon(y-z)\,dz\,dy\,ds.$$
$$+ \sup_{y\in\mathbb{R},s\leq T} \mathbb{E}\left[u(s,y)^2\right] \int_0^t \int \left(R_\varepsilon(s,x,z) - G(s,x,z)\right)^2 dz\,ds$$

The second term here converges to zero as $\varepsilon \to 0$ by part 3 of Lemma 1.2.1. The first term is bounded by

$$C \sup_{s\leq T,y,z\in\mathbb{R}} \left(\mathbb{E}\left[|u(s,y)-u(s,z)|^2\right] \chi_{\{\rho_\varepsilon(y-z)>0\}}\right).$$

By a calculation similar to that in the proof of Lemma 1.3.4 later, we see that for $h \in (0,1]$,

$$\sup_{s\leq T,0<|y-z|\leq h} \mathbb{E}\left[|u(s,y)-u(s,z)|^2\right] \leq C\left(\sup_{0<|y-z|\leq h} |f(y)-f(z)|^2 + h^{1/2}\right)$$

so $\sup_{t\leq T,y\in\mathbb{R}} \mathbb{E}\left[J_7^2(\varepsilon,t,x)\right]$ converges to zero as $\varepsilon \to 0$ by the uniform continuity of f.

$$\mathbb{E}\left[J_8^2(\varepsilon,t,x)\right] \leq C \sup_{y\in\mathbb{R},s\leq T} \mathbb{E}\left[u(s,y)^2 + v(s,y)^2\right] \times$$
$$\int_0^t \int \left(\int R_\varepsilon(s,z)\rho_\varepsilon(y-z)\,dz - G(s,y)\right)^2 dy\,ds$$

where, by applying Jensen's inequality,

$$\begin{aligned}
&\int_0^t \int \left(\int R_\varepsilon(s,z)\rho_\varepsilon(y-z)\,dz - G(s,y) \right)^2 dy\,ds \\
\le\;& C \int_0^t \int \int (R_\varepsilon(s,z) - G(s,z))^2 \rho_\varepsilon(y-z)\,dz\,dy\,ds \\
+\;& C \int_0^t \int \int (G(s,z) - G(s,y))^2 \rho_\varepsilon(y-z)\,dz\,dy\,ds.
\end{aligned}$$

The first term converges to zero by part 3 of Lemma 1.2.1. To show that the second term converges, we apply the Dominated Convergence Theorem in stages. By continuity of G,

$$\int (G(s,z) - G(s,y))^2 \rho_\varepsilon(y-z)\,dz$$

converges to zero as $\varepsilon \to 0$ for each $s > 0$ and $y \in \mathbb{R}$. It is straightforward to see that for each $s > 0$,

$$\int \sup_{\varepsilon \in (0,1]} \left(\int (G(s,z) - G(s,y))^2 \rho_\varepsilon(y-z)\,dz \right) dy < \infty$$

so by dominated convergence, we have, for each $s > 0$,

$$\lim_{\varepsilon \to 0} \int \int (G(s,z) - G(s,y))^2 \rho_\varepsilon(y-z)\,dz\,dy = 0.$$

Now

$$\begin{aligned}
&\int_0^t \sup_{\varepsilon \in (0,1]} \left(\int \int (G(s,z) - G(s,y))^2 \rho_\varepsilon(y-z)\,dz\,dy \right) ds \\
\le\;& \int_0^t \sup_{\varepsilon \in (0,1]} \left(\int \int \left(G(s,z)^2 + G(s,y)^2 \right) \rho_\varepsilon(y-z)\,dz\,dy \right) ds \\
=\;& 2 \int_0^t \int G(s,y)^2\,dy\,ds
\end{aligned}$$

which is finite, so $\sup_{t\le T, y\in\mathbb{R}} \mathbb{E}[J_8^2(\varepsilon,t,x)]$ also converges uniformly to zero as $\varepsilon \to 0$ by another application of dominated convergence.

$$\begin{aligned}
\mathbb{E}\left[J_3^2(\varepsilon,t,x)\right] &\le C\mathbb{E}\left[\left(\int_0^t \int R_\varepsilon(t-s,x,y)|v_\varepsilon(s,y)-v(s,y)|\,dy\,ds\right)^2\right] \\
&\le C\int_0^t \sup_{y\in\mathbb{R}} \mathbb{E}\left[|v_\varepsilon(s,y)-v(s,y)|^2\right]\left(\int R_\varepsilon(t-s,y)\,dy\right)^2 ds \\
&\le C\int_0^t \sup_{y\in\mathbb{R}} \mathbb{E}\left[|v_\varepsilon(s,y)-v(s,y)|^2\right] ds
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}J_6^2 &\le C\int_0^t \sup_{y\in\mathbb{R}} \mathbb{E}\left[|u_\varepsilon(s,y)-u(s,y)|^2 + |v_\varepsilon(s,y)-v(s,y)|^2\right] \times \\
&\qquad\qquad \int\int R_\varepsilon(t-s,z)^2 \rho_\varepsilon(y-z)\,dy\,dz\,ds \\
&\le C\int_0^t \frac{1}{\sqrt{t-s}} \sup_{y\in\mathbb{R}} \mathbb{E}\left[|u_\varepsilon(s,y)-u(s,y)|^2 + |v_\varepsilon(s,y)-v(s,y)|^2\right] ds
\end{aligned}$$

by part 1 of Lemma 1.2.1. Hence $\mathbb{E}[J_3^2 + J_6^2]$ is bounded by

$$C\int_0^t \frac{1}{\sqrt{t-s}} \sup_{y\in\mathbb{R}} \mathbb{E}\left[|u_\varepsilon(s,y)-u(s,y)|^2 + |v_\varepsilon(s,y)-v(s,y)|^2\right] ds.$$

We follow a similar procedure for u and get, for each $t \le T$,

$$\begin{aligned}
&\sup_{y\in\mathbb{R}} \mathbb{E}\left[|u_\varepsilon(t,y)-u(t,y)|^2 + |v_\varepsilon(t,y)-v(t,y)|^2\right] \le \\
&\qquad C\int_0^t \frac{1}{\sqrt{t-s}} \sup_{y\in\mathbb{R}} \mathbb{E}\left[|u_\varepsilon(s,y)-u(s,y)|^2 + |v_\varepsilon(s,y)-v(s,y)|^2\right] ds \\
&\qquad +H(\varepsilon)
\end{aligned}$$

where $H(\varepsilon)$ converges to zero as $\varepsilon \to 0$. Let

$$J(s) = \sup_{y\in\mathbb{R}} \mathbb{E}\left[|u_\varepsilon(t,y)-u(t,y)|^2 + |v_\varepsilon(t,y)-v(t,y)|^2\right].$$

Applying Gronwall's inequality gives (1.22). Finally, we relax the conditions on f and g. Given $f, g \in C_{tem}$, we choose sequences of bounded, uniformly continuous functions $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$: set $f_n(x) = f((x \wedge n) \vee (-n))$ and similarly for g_n. Then f_n and g_n converge in C_{tem} to f and g respectively. We have, for some constant C independent of n,

$$
\begin{aligned}
&\mathbb{E}\left[|u(t,x) - u_n(t,x)|^2 + |v(t,x) - v_n(t,x)|^2\right] \\
&\quad \leq C \int G(t,x,y)((f(y) - f_n(y))^2 + (g(y) - g_n(y))^2)\, dy \\
&\quad + C \int_0^t \int G(t-s,x,y) \mathbb{E}\left[|u(s,y) - u_n(s,y)|^2 + |v(s,y) - v_n(s,y)|^2\right]\, dy\, ds \\
&\quad + C \int_0^t \int G(t-s,x,y)^2 \mathbb{E}\left[|u(s,y) - u_n(s,y)|^2 + |v(s,y) - v_n(s,y)|^2\right]\, dy\, ds.
\end{aligned}
$$

We then apply (1.8) for some $\lambda > 0$, and apply Lemma 1.1.4 on

$$
F_n(s) = \sup_{y\in\mathbb{R}} \mathbb{E}\left[\left(|u(t,y) - u_n(t,y)|^2 + |v(t,y) - v_n(t,y)|^2\right) e^{-\lambda|y|}\right]
$$

to see that $\lim_{n\to\infty} F_n(t) = 0$ for each $t > 0$. It follows that the comparison holds for u and v. □

1.3 The Garsia lemma and increment bounds

Here, we estimate the expectation of the supremum of a solution over an interval. This will be necessary later both to check tightness and prove a continuity result with respect to the initial condition. The main tool for doing this is the following lemma due to Garsia, Rodemich and Rumsey and taken from Nualart [23].

Lemma 1.3.1 *Let $p, \Psi : \mathbb{R}_+ \to \mathbb{R}_+$ be continuous, strictly increasing functions such that $\Psi(0) = p(0) = 0$ and $\lim_{t\to\infty} \Psi(t) = \infty$. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be continuous and fix $\mathbf{x_0} \in \mathbb{R}^d$. Let $B = B_1(\mathbf{x_0}) := \{\mathbf{x} \in \mathbb{R}^d \ : \ ||\mathbf{x} - \mathbf{x_0}|| < 1\}$. Suppose*

$$\Gamma = \int_B \int_B \Psi\left(\frac{|\phi(\mathbf{x}) - \phi(\mathbf{y})|}{p(||\mathbf{x} - \mathbf{y}||)}\right) d\mathbf{y}\, d\mathbf{x} < \infty.$$

Then, for all $\mathbf{x}, \mathbf{y} \in B$,

$$|\phi(\mathbf{x}) - \phi(\mathbf{y})| \le 8 \int_0^{2||\mathbf{x}-\mathbf{y}||} \Psi^{-1}(C\Gamma u^{-2d})\, p(du)$$

where C is a constant depending only on d.

We apply the above result to prove the following. Note that since all norms on $\mathbb{R}^d$ are equivalent, it doesn't matter which norm we use on the denominator.

Lemma 1.3.2 *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a random continuous function. Then for $p > 0$, $\gamma \ge 4d$ there exists a constant $C = C(p, \gamma, d)$ such that for all $\mathbf{x_0} \in \mathbb{R}^d$ we have*

$$\mathbb{E}\left[\sup_{\mathbf{x},\mathbf{y}\in B, \mathbf{x}\neq\mathbf{y}} \frac{|\phi(\mathbf{x}) - \phi(\mathbf{y})|^p}{||\mathbf{x} - \mathbf{y}||^{\gamma/2}}\right] \le C \sup_{\mathbf{x},\mathbf{y}\in B, \mathbf{x}\neq\mathbf{y}} \frac{\mathbb{E}[|\phi(\mathbf{x}) - \phi(\mathbf{y})|^p]}{||\mathbf{x} - \mathbf{y}||^{\gamma}}$$

where $B = B_1(\mathbf{x_0})$. In particular,

$$\mathbb{E}\left[\sup_{\mathbf{x},\mathbf{y}\in B, \mathbf{x}\neq\mathbf{y}} |\phi(\mathbf{x}) - \phi(\mathbf{y})|^p\right] \le C \sup_{\mathbf{x},\mathbf{y}\in B, \mathbf{x}\neq\mathbf{y}} \frac{\mathbb{E}[|\phi(\mathbf{x}) - \phi(\mathbf{y})|^p]}{||\mathbf{x} - \mathbf{y}||^{\gamma}}.$$

Proof Set $\Psi(t) = t^p$ and $p(t) = t^{\gamma/p}$. Let

$$\Gamma = \int_B \int_B \frac{|\phi(\mathbf{x}) - \phi(\mathbf{y})|}{||\mathbf{x} - \mathbf{y}||^{\gamma}}\, d\mathbf{x}\, d\mathbf{y}.$$

Then

$$\mathbb{E}\Gamma \le |B|^2 \sup_{\mathbf{x},\mathbf{y}\in B, \mathbf{x}\neq\mathbf{y}} \frac{\mathbb{E}[|\phi(\mathbf{x}) - \phi(\mathbf{y})|^p]}{||\mathbf{x} - \mathbf{y}||^{\gamma}}.$$

We let $C = C(p,\gamma,d)$ be a constant whose value may change from line to line. By the previous lemma, for each $\mathbf{x}, \mathbf{y} \in B$,

$$
\begin{aligned}
|\phi(\mathbf{x}) - \phi(\mathbf{y})|^p &\leq C\Gamma\left(\int_0^{2\|\mathbf{x}-\mathbf{y}\|} u^{((\gamma-2d)/p)-1}\,du\right)^p \\
&\leq C\Gamma\|\mathbf{x}-\mathbf{y}\|^{\gamma-2d} \\
&\leq C\Gamma\|\mathbf{x}-\mathbf{y}\|^{\gamma/2}
\end{aligned}
$$

if $\gamma \geq 4d$. Rearranging and taking expectations gives the result. □

Suppose u is a C_{tem}-valued continuous solution of (1.1) with $u(0) = f$. We would like to be able to bound moments of $|u(t+k, x+h) - u(t,x)|$ for h, k in $(0,1]$ and use the above lemma to show that this gives us control on the expectation of the supremum of u over a space-time block. However, these increments may be badly behaved near $t = 0$ if f is rough, so to get around this problem we let $\overline{u}(t,x) = u(t,x) - \int G(t,x,y)f(y)\,dy$. By (1.4), $\overline{u}(t,x)$ is

$$
\int_0^t \int a(u(s,y))G(t-s,x,y)\,dy\,ds + \int_0^t \int b(u(s,y))G(t-s,x,y)\,W(dy\,ds)
$$

if a, b satisfy (1.3). We now bound moments of increments of $\overline{u}$ instead. The following deterministic lemma is key to this.

Lemma 1.3.3 *For all $t, \lambda, \theta \geq 0$ there exists a constant $C = C(t,\lambda,\theta)$ such that for all $x \in \mathbb{R}$ and $y_1, y_2 \in B_1(x)$,*

$$
\begin{aligned}
\int_0^t \int \left|G^\theta(s,y_1,z) - G^\theta(s,y_2,z)\right| e^{\lambda|z|}\,dz\,ds &\leq Ce^{\lambda|x|}|y_1 - y_2|, \\
\int_0^t \int \left|G^\theta(s,y_1,z) - G^\theta(s,y_2,z)\right|^2 e^{\lambda|z|}\,dz\,ds &\leq Ce^{\lambda|x|}|y_1 - y_2|
\end{aligned}
$$

and for $x \in \mathbb{R}$, $k \in [0,1]$,

$$\int_0^t \int \left|G^\theta(s+k,x,y) - G^\theta(s,x,y)\right| e^{\lambda|y|}\,dy\,ds \leq Ce^{\lambda|x|}k^{1/2},$$
$$\int_0^t \int \left|G^\theta(s+k,x,y) - G^\theta(s,x,y)\right|^2 e^{\lambda|y|}\,dy\,ds \leq Ce^{\lambda|x|}k^{1/2}.$$

Furthermore, if $8\lambda^2 < \theta$ then C is independent of t.

Proof

$$\begin{aligned} & e^{-\theta s}\left|e^{-(y_1-z)^2/4s} - e^{-(y_2-z)^2/4s}\right| e^{\lambda|z|} \qquad (1.23)\\ = \; & e^{-\theta s}\left|e^{-(y_1-z)^2/8s} - e^{-(y_2-z)^2/8s}\right|\left|e^{-(y_1-z)^2/8s} + e^{-(y_2-z)^2/8s}\right| e^{\lambda|z|}.\end{aligned}$$

By (1.7), for $y \in B_1(x)$,

$$e^{\lambda|z|} \leq e^{\lambda|y-z|}e^{\lambda(|x|+1)} \leq e^{(y-z)^2/8s}e^{8s\lambda^2}e^{\lambda(|x|+1)}$$

so (1.23) is bounded by

$$2e^{\lambda}e^{(8\lambda^2-\theta)s}e^{\lambda|x|}\left|e^{-(y_1-z)^2/8s} - e^{-(y_2-z)^2/8s}\right|.$$

Hence

$$\begin{aligned} & \int \left|G^\theta(s,y_1,z) - G^\theta(s,y_2,z)\right| e^{\lambda|z|}\,dz \\ & \leq e^{\lambda}e^{\lambda|x|}\frac{e^{(8\lambda^2-\theta)s}}{\sqrt{\pi s}}\int \left|e^{-(z+h)^2/8s} - e^{-z^2/8s}\right|\,dz, \qquad (1.24)\end{aligned}$$

where $h := |y_1 - y_2| \leq 2$. Since $e^{-(z+h)^2/8s} \geq e^{-z^2/8s}$ if and only if $z \leq -h/2$,

$$\begin{aligned} & \int \left|e^{-(z+h)^2/8s} - e^{-z^2/8s}\right|\,dz \\ = \; & \int_{-\infty}^{-h/2}\left(e^{-(z+h)^2/8s} - e^{-z^2/8s}\right)dz + \int_{-h/2}^{\infty}\left(e^{-z^2/8s} - e^{-(z+h)^2/8s}\right)dz\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_{-\infty}^{h/2} - \int_{-\infty}^{-h/2} + \int_{-h/2}^{\infty} - \int_{h/2}^{\infty}\right) e^{-z^2/8s}\,dz \\
&= 2\int_{-h/2}^{-h/2} e^{-z^2/8s}\,dz \leq 2h. \qquad (1.25)
\end{aligned}
$$

So, using (1.24) and (1.25),

$$
\int_0^t \int \left|G^\theta(s,y_1,z) - G^\theta(s,y_2,z)\right| e^{\lambda|z|}\,dy\,ds \;\leq\; 2he^\lambda e^{\lambda|x|}\int_0^t \frac{e^{(8\lambda^2-\theta)s}}{\sqrt{\pi s}}\,ds.
$$

This completes the proof of the first estimate. Similarly,

$$
\begin{aligned}
&e^{-2\theta s}\left|e^{-(y_1-z)^2/4s} - e^{-(y_2-z)^2/4s}\right|^2 e^{\lambda|z|} \\
&\qquad \leq\; 4e^\lambda e^{\lambda|x|} e^{(4\lambda^2-2\theta)s}\left|e^{-(y_1-z)^2/8s} - e^{-(y_2-z)^2/8s}\right|^2
\end{aligned}
$$

so

$$
\begin{aligned}
&\int_0^t \int \left|G^\theta(s,y_1,z) - G^\theta(s,y_2,z)\right|^2 e^{\lambda|z|}\,dz\,ds \\
&\leq\; e^\lambda e^{\lambda|x|}\int_0^t \frac{e^{(4\lambda^2-2\theta)s}}{\pi s}\int \left(e^{-(z+h)^2/8s} - e^{-z^2/8s}\right)^2 dz\,ds.
\end{aligned}
$$

By substituting $r = s/h^2$ and $w = z/h$, and setting $\alpha = (4\lambda^2 - 2\theta)_+$, we see that the above is bounded by

$$
he^{\lambda(|x|+1)}e^{\alpha t}\int_0^\infty \frac{1}{\pi r}\int \left(e^{-(w+1)^2/8r} - e^{-w^2/8r}\right)^2 dw\,dr.
$$

Since $\int (e^{-(w+1)^2/8r} - e^{-w^2/8r})^2\,dw \leq 2\int e^{-w^2/4r}\,dw = 4\sqrt{\pi r}$, it is clear that the integral for $r \in [0,1]$ converges. For $r \in [1,\infty)$, note that by the Mean Value Theorem,

$$
\left|e^{-(w+1)^2/8r} - e^{w^2/8r}\right|^2 \leq \sup_{u\in[w,w+1]} \frac{u^2 e^{-u^2/4r}}{16r^2} \leq \frac{(|w|+1)^2 e^{-w^2/16r} e^{1/4r}}{16r^2}.
$$

Now substitute $v = w/\sqrt{r}$. Then

$$\int_1^\infty \frac{1}{\pi r}\int \frac{(|w|+1)^2 e^{-w^2/16r} e^{1/4r}}{r^2}\,dw\,dr$$
$$= \int_1^\infty \frac{e^{1/4}}{16\pi r^{5/2}}\int (\sqrt{r}\,|v|+1)^2 e^{-v^2/16}\,dv\,dr$$

which converges, so we have the second estimate.

$$\int_0^t\int \left|G^\theta(s+k,x,z) - G^\theta(s,x,z)\right| e^{\lambda|z|}\,dz\,ds$$
$$\le \int_0^t\int e^{-\theta s}\frac{e^{-(x-z)^2/4(s+k)}}{\sqrt{4\pi(s+k)}}\left|e^{-\theta k}-1\right| e^{\lambda|z|}\,dz\,ds$$
$$+ \int_0^t\int e^{-\theta s}\frac{1}{\sqrt{4\pi(s+k)}}\left|e^{-(x-z)^2/4(s+k)} - e^{(x-z)^2/4s}\right| e^{\lambda|z|}\,dz\,ds$$
$$+ \int_0^t\int e^{-\theta s}e^{-(x-z)^2/4s}\left(\frac{\sqrt{s+k}-\sqrt{s}}{\sqrt{4\pi(s+k)s}}\right) e^{\lambda|z|}\,dz\,ds.$$

Let the three terms on the right be I_1, I_2 and I_3. Applying (1.7) as before,

$$I_1 \le \theta k e^{8\lambda^2} e^{\lambda|x|}\int_0^t \frac{e^{(8\lambda^2-\theta)s}}{\sqrt{4\pi(s+k)}}\int e^{-z^2/8(s+k)}\,dz\,ds$$
$$= \theta\sqrt{2}\,e^{8\lambda^2} k e^{\lambda|x|}\int_0^t e^{(8\lambda^2-\theta)s}\,ds.$$
$$I_2 \le \int_0^t\int \frac{2e^{-(x-z)^2/8(s+1)}e^{\lambda|z|}e^{-\theta s}}{\sqrt{4\pi(s+k)}}\left(e^{-(x-z)^2/8(s+k)} - e^{-(x-z)^2/8s}\right) dz\,ds$$
$$\le 2\sqrt{2}\,e^{8\lambda^2}e^{\lambda|x|}\int_0^t e^{(8\lambda^2-\theta)s}\int \frac{e^{-z^2/8(s+k)} - e^{-z^2/8s}}{\sqrt{8\pi(s+k)}}\,dy\,ds$$
$$\le 2\sqrt{2}\,e^{8\lambda^2}e^{\lambda|x|}\int_0^t \left(e^{(8\lambda^2-\theta)s}\right)\frac{\sqrt{s+k}-\sqrt{s}}{\sqrt{s+k}}\,ds$$
$$\le 2\sqrt{2k}\,e^{8\lambda^2}e^{\lambda|x|}\int_0^t \frac{k^{1/2}e^{(8\lambda^2-\theta)s}}{s+k}\,ds.$$

It is easy to see that the integral here is bounded independently of $k \in (0,1]$, and also bounded independently of t if $\theta > 8\lambda^2$.

$$\begin{aligned} I_3 &\le e^{\lambda|x|}\int_0^t e^{(8\lambda^2-\theta)s}\left(\frac{\sqrt{s+k}-\sqrt{s}}{\sqrt{4\pi s(s+k)}}\right)\int e^{-(x-z)^2/8s}\,dz\,ds \\ &\le \sqrt{2}\,ke^{\lambda|x|}\int_0^t \frac{e^{(8\lambda^2-\theta)s}}{s+k}\,ds = \sqrt{2k}\,e^{\lambda|x|}\int_0^t \frac{k^{1/2}e^{(8\lambda^2-\theta)s}}{s+k}\,ds \end{aligned}$$

which we can control in the same manner as for I_2.

$$\begin{aligned} &\frac{1}{9}\int_0^t\int \left|G^\theta(s+k,x,z)-G^\theta(s,x,z)\right|^2 e^{\lambda|z|}\,dz\,ds \\ &\le \int_0^t\int e^{-2\theta s}\frac{e^{-(x-z)^2/2(s+k)}}{4\pi(s+k)}\left|e^{-\theta k}-1\right|^2 e^{\lambda|z|}\,dz\,ds \\ &+ \int_0^t\int e^{-2\theta s}\frac{1}{4\pi(s+k)}\left|e^{-(x-z)^2/4(s+k)}-e^{-(x-z)^2/4s}\right|^2 e^{\lambda|z|}\,dz\,ds \\ &+ \int_0^t\int e^{-2\theta s}e^{-(x-z)^2/2s}\left(\frac{\sqrt{s+k}-\sqrt{s}}{\sqrt{4\pi(s+k)s}}\right)^2 e^{\lambda|z|}\,dz\,ds. \end{aligned}$$

Again, we label these terms I_1, I_2 and I_3 and apply a similar analysis.

$$\begin{aligned} I_1 &\le (\theta k)^2e^{4\lambda^2}e^{\lambda|x|}\int_0^t \frac{e^{(4\lambda^2-2\theta)s}}{4\pi(s+k)}\int e^{-z^2/4(s+k)}\,dz\,ds \\ &\le (\theta k)^2e^{4\lambda^2}e^{\lambda|x|}\int_0^t \frac{e^{(4\lambda^2-\theta)s}}{\sqrt{4\pi(s+k)}}\,ds. \\ I_2 &\le 2e^{4\lambda^2}e^{\lambda|x|}\int_0^t \frac{e^{(4\lambda^2-2\theta)s}}{4\pi(s+k)}\int\left(e^{-(x-z)^2/4(s+k)}-e^{-(x-z)^2/4s}\right)dz\,ds \\ &= (2/\sqrt{\pi})e^{4\lambda^2}e^{\lambda|x|}\int_0^t e^{(4\lambda^2-2\theta)s}\left(\frac{\sqrt{s+k}-\sqrt{s}}{s+k}\right)ds \\ &\le (2/\sqrt{\pi})\,e^{4\lambda^2}e^{\lambda|x|}k^{1/2}e^{\alpha t}\int_0^t \frac{k^{1/2}}{(s+k)^{3/2}}\,ds \\ &\le (2/\sqrt{\pi})\,e^{4\lambda^2}e^{\lambda|x|}k^{1/2}e^{\alpha t}\int_0^\infty \frac{1}{(r+1)^{3/2}}\,dr. \\ I_3 &\le k^{1/2}e^{\lambda|x|}e^{\alpha t}\int_0^t\left(\frac{k^{3/2}}{(s+k)^2\sqrt{4\pi s}}\right)ds \end{aligned}$$

$$\leq \quad k^{1/2} e^{\lambda|x|} e^{\alpha t} \int_0^\infty \left(\frac{1}{(r+1)^2\sqrt{4\pi r}} \right) dr$$

as required. □

We first deal with moments of increments in space.

Lemma 1.3.4 *Suppose a and b satisfy (1.3) and u is a C_{tem}-valued continuous solution of (1.1) with $u(0) = f \in C_{tem}$. Then for each $p \geq 2$, $\lambda \geq 0$ and $T \geq 0$ there exists a constant $C = C(p, \lambda, T)$ such that for all $x \in \mathbb{R}$ and $y_1, y_2 \in B_1(x)$,*

$$\begin{aligned} &\sup_{t\leq T} \mathbb{E}\left[|\overline{u}(t,y_1) - \overline{u}(t,y_2)|^p|\right] \\ &\quad \leq \quad C|y_1-y_2|^{p/2} e^{\lambda|x|} \sup_{s\leq T, y\in\mathbb{R}} \mathbb{E}\left[(|a(u(s,y))| + |b(u(s,y))|)^p e^{-\lambda|y|}\right]. \end{aligned}$$

Proof For some constant $C = C(p)$, using Burkholder's inequality,

$$\begin{aligned} &\mathbb{E}[|\overline{u}(t,y_1) - \overline{u}(t,y_2)|^p] \\ &\quad\leq \quad C\mathbb{E}\left[\left|\int_0^t \int a(u(s,y))\left(G(t-s,y_1,z) - G(t-s,y_2,z)\right) dz\, ds\right|^p\right] \\ &\quad + \quad C\mathbb{E}\left[\left|\int_0^t \int b(u(s,y))^2 \left(G(t-s,y_1,z) - G(t-s,y_2,z)\right)^2 dz\, ds\right|^{p/2}\right] \Bigg\} \\ &\quad\leq \quad C \sup_{s\leq t, z\in\mathbb{R}} \mathbb{E}\left[(|a(u(s,z))|^p + |b(u(s,y))|^p) e^{-\lambda|z|}\right] \times \\ &\qquad\qquad \Bigg\{ \left(\int_0^t \int |G(s,y_1,z) - G(s,y_2,z)|\, e^{\lambda|z|/p} dz\, ds \right)^p + \\ &\qquad\qquad \left(\int_0^t \int |G(s,y_1,z) - G(s,y_2,z)|^2 e^{2\lambda|z|/p} dz\, ds \right)^{p/2} \Bigg\}. \end{aligned}$$

The result now follows from Lemma 1.3.3. □

We now prove a similar estimate for time increments.

Lemma 1.3.5 *Let u be as in Lemma 1.3.4. For each $p \geq 2$, $\lambda \geq 0$ and $T \geq 0$ there exists a constant $C = C(p, \lambda, T)$ such that for all $t \leq T$, $x \in \mathbb{R}$ and $k \in [0, 1]$,*

$$\begin{aligned}
&\mathbb{E}\left[|\overline{u}(t+k,x) - \overline{u}(t,x)|^p\right] \\
&\qquad \leq Ck^{p/4}e^{\lambda|x|} \sup_{s \leq t+k, y \in \mathbb{R}} \mathbb{E}\left[\left(|a(u(s,y))| + |b(u(s,y))|\right)^p e^{-\lambda|y|}\right].
\end{aligned}$$

Proof For some $C = C(p)$,

$$\begin{aligned}
&\mathbb{E}\left[|\overline{u}(t+k,x) - \overline{u}(t,x)|^p\right] \\
&\leq C\mathbb{E}\left[\left|\int_0^t \int a(u(s,y))\left(G(t+k-s,x,y) - G(t-s,x,y)\right) dy\, ds\right|^p\right] \\
&+ C\mathbb{E}\left[\left|\int_t^{t+k} \int a(u(s,y))G^\theta(t+k-s,x,y)\, dy\, ds\right|^p\right] \\
&+ C\mathbb{E}\left[\left|\int_0^t \int b(u(s,y))^2 \left(G(t+k-s,x,y) - G(t-s,x,y)\right)^2 dy\, ds\right|^{p/2}\right] \\
&+ C\mathbb{E}\left[\left|\int_t^{t+k} \int b(u(s,y))^2 G(t+k-s,x,y)^2\, dy\, ds\right|^{p/2}\right] \\
&\leq C \sup_{s \leq t+k, y \in \mathbb{R}} \mathbb{E}\left[\left(|a(u(s,y))|^p + |b(u(s,y))|^p\right) e^{-\lambda|y|}\right] \times \\
&\qquad \left\{ \left(\int_0^t \int |G(s+k,x,y) - G(s,x,y)|\, e^{\lambda|y|/p}\, dy\, ds\right)^p + \right. \\
&\qquad \left(\int_0^t \int |G(s+k,x,y) - G(s,x,y)|^2\, e^{2\lambda|y|/p}\, dy\, ds\right)^{p/2} + \\
&\qquad \left(\int_0^k \int G(s,x,y)e^{\lambda|y|/p}\, dy\, ds\right)^p + \\
&\qquad \left.\left(\int_0^k \int G(s,x,y)^2 e^{2\lambda|y|/p}\, dy\, ds\right)^{p/2} \right\}
\end{aligned}$$

which we can control by applying Lemma 1.3.3 and (1.8). □

Note that if $\lambda = 0$, the bounds we get from the above two lemmas are infinite unless f is bounded. Putting Lemmas 1.3.4 and 1.3.5 together, we see that for each $p \geq 2$, $\lambda \geq 0$ and $T > 0$, there is a constant $C = C(p, \lambda, L, T)$ such that for all $x \in \mathbb{R}$, $y_1, y_2 \in B_1(x)$ and $0 \leq t_1, t_2 \leq T$ with $0 < |t_1 - t_2| \leq 1$, we have

$$\begin{aligned} &\mathbb{E}\left[|\overline{u}(t_1, y_1) - \overline{u}(t_2, y_2)|^p\right] \\ &\quad \leq \; Ce^{\lambda|x|}\left(|y_1 - y_2|^{p/2} + |t_1 - t_2|^{p/4}\right) \sup_{s \leq T, y \in \mathbb{R}} \mathbb{E}\left[|u(s, y)|^p e^{-\lambda|y|} + 1\right]. \end{aligned}$$

Applying Lemma 1.1.3, we can bound this by

$$Ce^{\lambda|x|}\left(|y_1 - y_2|^{p/2} + |t_1 - t_2|^{p/4}\right) \sup_{y \in \mathbb{R}}\left(|f(y)|^p e^{-\lambda|y|} + 1\right) \tag{1.26}$$

for some possibly different constant C.

Corollary 1.3.6 *Suppose u is as in Lemma 1.3.4. Then for all $T \geq 0$, $p \geq 32$ and $\lambda \geq 0$, there exists a constant $C = C(p, \lambda, L, T)$ such that for all $x \in \mathbb{R}$,*

$$\begin{aligned} &\sup_{t \leq T} \mathbb{E}\left[\sup_{(t_1, y_1), (t_2, y_2) \in B_1((t, x))} \frac{|\overline{u}(t_1, y_1) - \overline{u}(t_2, y_2)|^p}{\|(t_1, y_1) - (t_2, y_2)\|^{p/8}}\right] \\ &\qquad \leq \; Ce^{\lambda|x|} \sup_{y \in \mathbb{R}}\left(|f(y)|^p e^{-\lambda|y|} + 1\right) \end{aligned} \tag{1.27}$$

and for all $\delta \in (0, 1]$,

$$\begin{aligned} &\sup_{t \leq T} \mathbb{E}\left[\sup_{(t_1, y_1), (t_2, y_2) \in B_\delta((t, x))} |\overline{u}(t_1, y_1) - \overline{u}(t_2, y_2)|^p\right] \\ &\qquad \leq \; C\delta^{p/8} e^{\lambda|x|} \sup_{y \in \mathbb{R}}\left(|f(y)|^p e^{-\lambda|y|} + 1\right). \end{aligned} \tag{1.28}$$

Proof We get (1.27) immediately by applying Lemma 1.3.2 to (1.26). Since

$$\begin{aligned} &\mathbb{E}\left[\sup_{(t_1, y_1), (t_2, y_2) \in B_\delta((t, x))} |\overline{u}(t_1, y_1) - \overline{u}(t_2, y_2)|^p\right] \\ &\qquad \leq \; \delta^{p/8} \mathbb{E}\left[\sup_{(t_1, y_1), (t_2, y_2) \in B_1((t, x))} \frac{|\overline{u}(t_1, y_1) - \overline{u}(t_2, y_2)|^p}{\|(t_1, y_1) - (t_2, y_2)\|^{p/8}}\right], \end{aligned}$$

(1.28) follows. □

We need the following tightness condition for Section 1.8.

Lemma 1.3.7 *Suppose u is as in Lemma 1.3.4 and for all $p \geq 2$ and $\lambda > 0$,*

$$\sup_{t \geq 0, y \in \mathbb{R}} \mathbb{E}\left[|u(s,y)|^p e^{-\lambda|y|}\right] < \infty.$$

Then for all $\delta > 0$, the family of distributions $\{\mu_t\}_{t \geq \delta}$ on C_{tem} induced by $u(t)$ for $t \geq \delta$ is tight.

Proof We check the tightness condition in Proposition 1.9.2. Set

$$\overline{u}^\theta(t,x) = u(t,x) - \int G^\theta(t,x,y) f(y)\, dy.$$

By Lemma 1.1.7, this is equal to

$$\int_0^t \int G^\theta(t-s,x,y)\tilde{a}(u(s,y))\, dy\, ds + \int_0^t \int G^\theta(t-s,x,y) b(u(s,y))\, W(dy\, ds)$$

where $\tilde{a}(u) = a(u) + \theta u$. By a similar calculation as in the proof of Lemma 1.3.4, and applying Lemma 1.3.3, we see that for each $p \geq 2$, $\lambda > 0$ and $\theta > 8\lambda^2$ there exists a constant $C = C(p,\lambda,\theta,L)$ such that for all $x \in \mathbb{R}$, $T \geq 0$ and $y_1, y_2 \in B_1(x)$,

$$\begin{aligned}
&\sup_{t \leq T} \mathbb{E}\left[\left|\overline{u}^\theta(t,y_1) - \overline{u}^\theta(t,y_2)\right|^p\right] \\
&\quad \leq C|y_1 - y_2|^{p/2} e^{\lambda|x|} \sup_{t \leq T, y \in \mathbb{R}} \mathbb{E}\left[\left(|u(t,y)|^p + 1\right) e^{-\lambda|y|}\right].
\end{aligned}$$

In particular, C does not depend on T. So, for all $t \geq 0$,

$$\begin{aligned}
&\mathbb{E}\left[|u(t,y_1) - u(t,y_2)|^p\right] \\
&\quad \leq 2^p \left(\int |f(z)| \left|G^\theta(t,y_1,z) - G^\theta(t,y_2,z)\right| dz\right)^p \\
&\quad + 2^p C|y_1 - y_2|^{p/2} e^{\lambda|x|} \sup_{s \geq 0, y \in \mathbb{R}} \mathbb{E}\left[\left(|u(s,y)|^p + 1\right) e^{-\lambda|y|}\right].
\end{aligned}$$

By (1.24) and (1.25), the first term here is bounded by

$$C\left(\sup_{z\in\mathbb{R}}|f(z)|^p e^{-\lambda|z|}\right)\delta^{-1/2}e^{\lambda|x|}|y_1-y_2|^p$$

for some $C=C(p,\lambda)$ if $t\geq\delta$. Hence the tightness condition holds. □

1.4 Continuity with respect to the initial condition

Proposition 1.4.1 *Suppose a and b are Lipschitz, and $\{f_n\}_{n\in\mathbb{N}}\subset C_{tem}$ is a sequence converging to $f\in C_{tem}$. Let $\{u_n\}_{n\in\mathbb{N}}$ and u be the C_{tem}-valued continuous solutions of (1.1) (with respect to the same white noise) started from $\{f_n\}$ and f respectively. Then u_n converges in probability to u, i.e. for each $T>0$, $\lambda>0$ and $\varepsilon>0$,*

$$\lim_{n\to\infty}\mathbb{P}\left(\sup_{s\leq T,y\in\mathbb{R}}|u_n(s,y)-u(s,y)|e^{-\lambda|y|}\geq\varepsilon\right)=0.$$

We prove this result by first splitting up $[0,T]\times\mathbb{R}$ into rectangles of the form $[0,T]\times[m,m+1]$ for $m\in\mathbb{Z}$, then further splitting each of these rectangles into $2^{2N}T$ squares with sides of length 2^{-N}. The expected variation of u and u_n over each of the squares will be small by our modulus of continuity results, and we control $|u(s,y)-u_n(s,y)|$ at a fixed point by the following lemma.

Lemma 1.4.2 *For all $\lambda>0$, $T>0$ and $p\geq 2$,*

$$\lim_{n\to\infty}\sup_{s\leq T,y\in\mathbb{R}}\mathbb{E}\left[|u(s,y)-u_n(s,y)|^p e^{-\lambda|y|}\right]=0.$$

Proof By the Green's function representation, we see that, for some constant C depending on p, T and the Lipschitz constant,

$$\begin{aligned}
&\mathbb{E}[|u(t,x) - u_n(t,x)|^p] \\
&\qquad \le \; C \int |f(y) - f_n(y)|^p G(t,x,y)\,dy \\
&\qquad + \; C \int_0^t \int \mathbb{E}[|u(s,y) - u_n(s,y)|^p]\, G(t-s,x,y)\,dy\,ds \\
&\qquad + \; C \int_0^t \int \mathbb{E}[|u(s,y) - u_n(s,y)|^p]\, G(t-s,x,y)^2\,dy\,ds.
\end{aligned}$$

Applying (1.8), we see that, for a constant $C = C(p, T, \lambda)$,

$$\begin{aligned}
&\sup_{y\in\mathbb{R}} \mathbb{E}\left[|u(t,y) - u_n(t,y)|^p e^{-\lambda|y|}\right] \\
&\qquad \le \; C \sup_{y\in\mathbb{R}} \left(|f(y) - f_n(y)|^p e^{-\lambda|y|}\right) \\
&\qquad + \; C \int_0^t \frac{1}{\sqrt{t-s}} \sup_{y\in\mathbb{R}} \mathbb{E}\left[|u(s,y) - u_n(s,y)|^p e^{-\lambda|y|}\right] ds
\end{aligned}$$

so the result follows by Lemma 1.1.4. □

Proof (of 1.4.1) Because $\int G(t,\cdot,y) f_n(y)\,dy$ converges to $\int G(t,\cdot,y) f(y)\,dy$ in the d_{tem} metric, it suffices to prove that for each $T \in \mathbb{N}$, $\lambda > 0$ and $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\left(\sup_{t\le T, x\in\mathbb{R}} |\overline{u}_n(s,x) - \overline{u}(s,x)| e^{-\lambda|x|} \ge \varepsilon \right) = 0.$$

Fix $m \in \mathbb{Z}$ and set

$$\square(j,k) = [2^{-N}j, 2^{-N}(j+1)] \times [m + 2^{-N}k, m + 2^{-N}(k+1)]$$

for $j = 0, \ldots, 2^N T - 1$ and $k = 0, \ldots, 2^N - 1$.

$$\mathbb{P}\left(\sup_{s\in[0,T], x\in[m,m+1]} |\overline{u}_n(s,x) - \overline{u}(s,x)| e^{-\lambda|m|} \ge \varepsilon \right)$$

$$
\begin{aligned}
&\leq \sum_{j=0}^{2^N T-1} \sum_{k=0}^{2^N-1} \mathbb{P}\left(\sup_{(s,x)\in\square(j,k)} \left|\overline{u}_n(s,x) - \overline{u}_n(2^{-N}j, m+2^{-N}k)\right| e^{-\lambda|m|} \geq \varepsilon/3 \right) \\
&+ \sum_{j=0}^{2^N T-1} \sum_{k=0}^{2^N-1} \mathbb{P}\left(\sup_{(s,x)\in\square(j,k)} \left|\overline{u}(s,x) - \overline{u}(2^{-N}j, m+2^{-N}k)\right| e^{-\lambda|m|} \geq \varepsilon/3 \right) \\
&+ 2^{2N} T \sup_{s\in[0,T]} \sup_{x\in[m,m+1]} \mathbb{P}\left(\left|\overline{u}_n(s,x) - \overline{u}(s,x)\right| e^{-\lambda|m|} \geq \varepsilon/3 \right).
\end{aligned}
$$

Label the three terms here by $I_1(N,m,n)$, $I_2(N,m)$ and $I_3(N,m,n)$ respectively. Applying Markov's inequality, we see that the first term is bounded by

$$
(3/\varepsilon)^p 2^{2N} T e^{-p\lambda|m|} \sup_{j,k} \mathbb{E}\left[\sup_{(s,x)\in\square(j,k)} \left|\overline{u}_n(s,x) - \overline{u}_n(2^{-N}j, m+2^{-N}k)\right|^p \right].
$$

Applying (1.28) we see that for $p = 32$,

$$
\sup_{j,k} \mathbb{E}\left[\sup_{(s,x)\in\square(j,k)} \left|\overline{u}_n(s,x) - \overline{u}_n(2^{-N}j, m+2^{-N}k)\right|^p \right] \leq 2^{-4N} C e^{\lambda|m|}
$$

for some constant C independent of m, N and n, so

$$
I_1(N,m,n) \leq (3/\varepsilon)^p 2^{-2N} T C e^{-31\lambda|m|}
$$

and hence

$$
\sup_{n\in\mathbb{N}} \sum_{m\in\mathbb{Z}} I_1(N,m,n) \leq 2^{-2N} C
$$

for some C independent of N so we can choose N large enough to make this as small as desired. The case for I_2 is similar. By Lemma 1.4.2,

$$
\lim_{n\to\infty} \sup_{s\in[0,T],x\in\mathbb{R}} \mathbb{E}\left[|\overline{u}_n(s,x) - \overline{u}(s,x)|^2 e^{-\lambda|x|} \right] = 0.
$$

Applying Markov's inequality again, we see that

$$
\lim_{n\to\infty} \sum_{m\in\mathbb{Z}} I_3(N,m,n) = 0
$$

for each fixed N. $\square$

Since convergence in probability implies weak convergence, we have the following corollary.

Corollary 1.4.3 *If a and b are Lipschitz and for each $f \in C_{tem}$, $\mathbb{P}_f$ is the probability distribution generated by a solution of (1.1) with initial condition f, then for each bounded continuous $H : C(\mathbb{R}_+ \to C_{tem}) \to \mathbb{R}$, the map*

$$f \mapsto \int H(\omega)\, \mathbb{P}_f(d\omega)$$

is continuous.

1.5 The non-Lipschitz case

In general, we may want to be able to compare two solutions where the coefficients are not Lipschitz. We consider here two particular cases necessary for Chapters 2 and 3. The first is the following.

Proposition 1.5.1 *Suppose $f, g \in C_{tem}$ and $f \geq g$. Suppose a and b are Lipschitz and $1/L \leq b \leq L$ for some constant $L > 0$. Suppose further that*

$$\alpha(U) = \sqrt{\frac{(K - |U|)^+}{K}}, \qquad \beta(U) = \sqrt{\frac{|U| \wedge K}{K}}.$$

Then there exists a filtered probability space with a pair of continuous, C_{tem}-valued processes (u, v) and a pair of independent white noises W_1 and W_2 such that (u, v) is a solution of the system

$$\left.\begin{aligned} \partial_t u &= \Delta u + a(u) + b(u)\, dW_1 \\ \partial_t v &= \Delta v + a(v) + b(v)(\alpha(u - v)\, dW_1 + \beta(u - v)\, dW_2) \end{aligned}\right\} \quad (1.29)$$

with $(u(0), v(0)) = (f, g)$*, and* $u \geq v$ *almost surely.*

Proof The upper bound on b along with the Lipschitz assumption on a and b ensures that (1.20) is satisfied. Taking b bounded away from zero means we can avoid a degeneracy problem when we come to construct the required noises later.

We choose sequences of Lipschitz functions α_n and β_n converging uniformly to α and β respectively, and such that $\alpha^2 + \beta^2 \equiv 1$. By the comparison result given earlier, for each $n \in \mathbb{N}$ we can construct a pair of solutions $(\tilde{u}_n, \tilde{v}_n)$ such that $(\tilde{u}_n(0), \tilde{v}_n(0)) = (f, g)$, $\tilde{u}_n \geq \tilde{v}_n$ almost surely, and $(\tilde{u}_n, \tilde{v}_n)$ satisfies

$$\begin{aligned} \partial_t \tilde{u}_n &= \Delta \tilde{u}_n + a(\tilde{u}_n) + b(\tilde{u}_n)\, d\tilde{W}_1 \\ \partial_t \tilde{v}_n &= \Delta \tilde{v}_n + a(\tilde{v}_n) + b(\tilde{v}_n)(\alpha_n(\tilde{U}_n)\, d\tilde{W}_1 + \beta_n(\tilde{U}_n)\, d\tilde{W}_2), \end{aligned}$$

where $\tilde{W}_1$ and $\tilde{W}_2$ are independent white noises and $\tilde{U}_n = \tilde{u}_n - \tilde{v}_n$. Note that since $\alpha_n^2 + \beta_n^2 \equiv 1$, $\alpha(\tilde{U}_n)\tilde{W}_1 + \beta(\tilde{U}_n)\tilde{W}_2$ is a white noise for each n. Hence each v_n is a solution of the same SPDE with respect to a different white noise and so by uniqueness in law, all the $\tilde{v}_n$ have the same distribution. By pathwise uniqueness, all the $\tilde{u}_n$ are equal almost surely.

For each n in $\mathbb{N}$, let $\tilde{\mathbb{P}}_n$ be the probability measure on $C(\mathbb{R}^+ \to C_{tem})^2$ generated by $(\tilde{u}, \tilde{v}_n)$. We equip $C(\mathbb{R}^+ \to C_{tem})$ with the metric given earlier, and $C(\mathbb{R}^+ \to C_{tem})^2$ with a product metric. By the increment bounds given in Lemmas 1.3.4 and 1.3.5, and since the distributions of u_n and v_n are independent of n, we see that the tightness condition given in Section 1.9 holds for $\{\tilde{u} - G(\cdot)f\}_{n \in \mathbb{N}}$ and $\{\tilde{v}_n - G(\cdot)g\}_{n \in \mathbb{N}}$. Hence $\{\tilde{\mathbb{P}}_n\}_{n \in \mathbb{N}}$ is also tight, so passing to a subsequence if necessary, there exists a probability measure $\tilde{\mathbb{P}}$ on $C(\mathbb{R}^+ \to C_{tem})^2$ such that $\tilde{\mathbb{P}}_n$ converges weakly to $\tilde{\mathbb{P}}$.

Since $C(\mathbb{R}^+ \to C_{tem})^2$ is separable, we can apply Skorohod's Theorem (see, for example Theorem 1.8 on page 102 of Ethier and Kurtz [8]) which gives us a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with pairs of continuous, C_{tem}-valued processes $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ and (u, v) such that (u_n, v_n) converges almost surely to (u, v). In particular, u_n and v_n converge uniformly to u and v respectively on each compact subset of $\mathbb{R}^+ \times \mathbb{R}$, almost surely. We equip this probability space with the filtration

$$\mathcal{F}_t = \sigma(\{(u|_{[0,t]}, v|_{[0,t]})^{-1}(A) : A \in \mathcal{B}(C([0,t] \to C_{tem})^2)\}).$$

It is not difficult to see that

$$\mathcal{F}_t = \sigma(\{(u(s), v(s))^{-1}(A) : s \le t, A \in \mathcal{B}(C^2_{tem})\}).$$

Similarly, for each $n \in \mathbb{N}$ we set

$$\mathcal{F}^n_t = \sigma(\{(u_n|_{[0,t]}, v_n|_{[0,t]})^{-1}(A) : A \in \mathcal{B}(C([0,t] \to C_{tem})^2)\}).$$

Note that $u(s,x)$ and $v(s,x)$ are $\mathcal{F}_t$ measurable for $s \le t$, $x \in \mathbb{R}$, and similarly for u_n and v_n.

To show that (u, v) solves the required system with respect to some pair of white noises, it will be convenient to work with *martingale problems* as in Ethier and Kurtz [8] and Rogers and Williams [28]. For $\phi \in C_c^\infty(\mathbb{R})$, set

$$M^\phi(t) = (u(t),\phi) - (f,\phi) - \int_0^t (u(s), \Delta\phi)\,ds - \int_0^t (a(u(s,\cdot)),\phi)\,ds \quad (1.30)$$

$$N^\phi(t) = (v(t),\phi) - (g,\phi) - \int_0^t (v(s), \Delta\phi)\,ds - \int_0^t (a(v(s,\cdot)),\phi)\,ds \quad (1.31)$$

and for each $n \in \mathbb{N}$, set

$$M^\phi_n(t) = (u_n(t),\phi) - (f,\phi) - \int_0^t (u_n(s), \Delta\phi)\,ds - \int_0^t (a(u_n(s,\cdot)),\phi)\,ds \quad (1.32)$$

$$N^\phi_n(t) = (v_n(t),\phi) - (g,\phi) - \int_0^t (v_n(s), \Delta\phi)\,ds - \int_0^t (a(v_n(s,\cdot)),\phi)\,ds. \quad (1.33)$$

Since $u_n, v_n, \phi, \Delta\phi, f, g, a$ are all continuous, the above integrals are defined in the Riemann sense and so by approximating the integrals with Riemann sums, we see that $M_n^\phi(t)$ and $N_n^\phi(t)$ are $\mathcal{F}_t^n$-measurable. Similarly, $M^\phi(t)$ and $N^\phi(t)$ are $\mathcal{F}_t$-measurable. Since each (u_n, v_n) is equal in distribution to $(\tilde{u}_n, \tilde{v}_n)$, it follows that M_n^ϕ and N_n^ϕ are $\mathcal{F}_t^n$-martingales and that the quadratic variations are

$$\begin{aligned}
\langle M_n^\phi \rangle_t &= \int_0^t \int b(u_n(s,x))^2 \phi(x)^2 \, dx \, ds \\
\langle N_n^\phi \rangle_t &= \int_0^t \int b(v_n(s,x))^2 \phi(x)^2 \, dx \, ds \\
\langle M_n^\phi, N_n^\psi \rangle_t &= \int_0^t \int b_n(u_n(s,x)) b_n(v_n(s,x)) \phi(x) \psi(x) \alpha_n(U_n(s,x)) \, dx \, ds.
\end{aligned}$$

We now need to show that M^ϕ and N^ϕ are $\mathcal{F}_t$-martingales with quadratic variations analogous to those of M_n^ϕ and N_n^ϕ. For each s, t with $0 < s < t$, and each $A \in \mathcal{F}_s^n$,

$$\mathbb{E}\left[(M_n^\phi(t) - M_n^\phi(s)) 1_A\right] = 0$$

by definition of a martingale. Approximating by linear combinations of indicator functions, we see that if $h : C(\mathbb{R}^+ \to C_{tem})^2 \to \mathbb{R}$ is a bounded continuous function such that $h(u, v)$ depends only on $(u|_{[0,s]}, v|_{[0,s]})$, then

$$\mathbb{E}\left[(M_n^\phi(t) - M_n^\phi(s)) h(u_n, v_n)\right] = 0.$$

To show that $\mathbb{E}\left[(M^\phi(t) - M^\phi(s)) h(u, v)\right] = 0$, it suffices to check that for each $t > 0$, $\{M_n^\phi(t)\}_{n \in \mathbb{N}}$ is uniformly integrable and $M_n^\phi(t)$ converges to $M^\phi(t)$ almost surely as $n \to \infty$. To show uniform integrability, we show $\mathbb{E}\left[M^\phi(t)^p\right]$ is bounded independently of n for $p \geq 1$: applying Hölder's inequality gives

$$\mathbb{E}\left[\left(\int_0^t \int a(u_n(s,x)) \phi(x) \, dx \, ds\right)^p\right]$$

$$\leq \mathbb{E}\left[\int_0^t \int |a(u_n(s,x))|^p |\phi(x)|\,dx\,ds\right]\left(\int_0^t \int |\phi(x)|\,dx\,ds\right)^{p-1}$$
$$\leq \left(\int \int |\phi(x)|\,dx\,ds\right)^p \sup_{s\leq t, x\in A} \mathbb{E}[|a(u_n(s,y))|^p]$$

where $A = \operatorname{supp}\phi$. This is bounded independently of n by Lemma 1.1.3 since the distributions of u_n are equal. The other terms in (1.32) are dealt with in the same way. Note that this shows that higher powers of $M_n^\phi(t)$ are also uniformly integrable.

Almost sure convergence follows since ϕ is compactly supported and u_n converges uniformly to u on compact sets. To complete the proof that M^ϕ is a martingale, we need to show that

$$\mathbb{E}\left[(M^\phi(t) - M^\phi(s))h(u,v)\right] = 0 \tag{1.34}$$

for indicator functions $h = \chi_A$, where $A \in \mathcal{F}_s$. We do this by applying the Monotone Class Theorem as follows. Let $\mathcal{H}$ be the class of functions bounded between 0 and 1, and depending only on $(u|_{[0,s]}, v_{[0,s]})$, for which (1.34) holds. Then $\mathcal{H}$ contains the continuous functions and is closed under pointwise convergence by dominated convergence, so by Corollary 4.4 on page 497 of Ethier and Kurtz [8], (1.34) holds for the required indicator functions. Hence M^ϕ is a martingale.

We also have

$$\mathbb{E}\left[\left(M_n^\phi(t)^2 - M_n^\phi(s)^2 - \int_s^t \int b(u_n(s,x))^2\phi(x)^2\,dy\,ds\right)h(u_n,v_n)\right] = 0$$

so, since $\{M_n^\phi(t)^2\}_{n\in\mathbb{N}}$ is also uniformly integrable, we can pass to the limit in n and apply the Monotone Class Theorem in the same way as above to show that

$$\langle M^\phi \rangle_t = \int_0^t \int b(u(s,x))^2\phi(x)^2\,dx\,ds.$$

An identical argument shows that $N^{\phi}(t)$ is also a martingale with quadratic variation $\int_0^t \int b(v(s,x))^2 \phi(x)^2 \, dx \, ds$.

Finally, we check that the covariance $\langle M^{\phi}, N^{\psi} \rangle_t$ takes the required value by similar means: uniform integrability of $\{M_n^{\phi} N_n^{\phi}(t)\}_{n \in \mathbb{N}}$ follows easily by Cauchy-Schwartz, so it is sufficient to check that, for $U = u - v$ and $U_n = u_n - v_n$,

$$\int_0^t \int_A |b(u_n(s,x))b(v_n(s,x))\alpha_n(U_n(s,x)) - b(u(s,x))b(v(s,x))\alpha(U(s,x))| \, dx \, dr$$

converges almost surely to zero for compact sets $A \subset \mathbb{R}$. This follows from the uniform convergence of α_n.

Although showing that (u, v) satisfies the right martingale problems gives us much information about the system, we would really like to show that u and v are actually solutions of (1.1) by constructing the appropriate white noises. We now show how to do this.

We first need to construct martingale measures M_t and N_t on $\mathbb{R}$ from the martingales M^{ϕ} and N^{ϕ}. Let $\mathcal{A}$ be the class of bounded Borel subsets of $\mathbb{R}$. Given $A \in \mathcal{A}$, we choose a sequence of functions $\phi_n \in C_c^{\infty}(\mathbb{R})$ converging to χ_A in L^2, and such that $\sup_n |\phi_n|$ is compactly supported and bounded.

$$\mathbb{E}\left[(M^{\phi_m}(t) - M^{\phi_n}(t))^2\right] \leq L^2 t \int (\phi_n(x) - \phi_m(x))^2 \, dx,$$

so by completeness, $M^{\phi_n}(t)$ has a limit in L^2 which we will call $M_t(A)$. It is straightforward to check that $M_t(A)$ is independent of the choice of ϕ_n. To show it is a martingale, we observe that since each M^{ϕ_n} is a martingale, for any bounded, $\mathcal{F}_s$-measurable function $h = h(u, v)$,

$$\mathbb{E}[(M_t(A) - M_s(A))h(u,v)]^2 =$$

$$\mathbb{E}\left[\left(\left(M_t(A) - M^{\phi_n}(t)\right) - \left(M_s(A) - M^{\phi_n}(s)\right)\right) h(u,v)\right]^2 \leq$$
$$(4 \sup |h|) \left(\mathbb{E}\left[(M_t(A) - M^{\phi_n}(t))^2\right] + \mathbb{E}\left[(M_t(A) - M^{\phi_n}(t))^2\right]\right)$$

which converges to zero as $n \to 0$. By a similar method, $M_t(A)$ has quadratic variation

$$\langle M(A)\rangle_t = \int_0^t \int_A b(u(s,x))^2 \, dx \, ds.$$

Following the procedure in Chapter 2 of Walsh [32], we can extend M to an orthogonal martingale measure with covariance measure

$$Q_M(dx\,dy\,ds) = b(u(s,x))^2 \delta_{\{x=y\}} \, dx\,dy\,ds.$$

The construction for N is similar. We also note that

$$\begin{aligned} \langle M(A), N(B)\rangle_t &= \int_0^t \int_{A\cap B} b(u(s,x)) b(v(s,x)) \alpha(U(s,x)) \, dx \, ds \\ &= \int_0^t \int_A \int_B Q_{M,N}(dx\,dy\,ds) \end{aligned}$$

where

$$Q_{M,N}(dx\,dy\,ds) = b(u(s,x)) b(v(s,x)) \alpha(U(s,x)) \delta_{\{x=y\}} \, dx\,dy\,ds.$$

Recall from Chapter 2 of Walsh [32] or the appendix that if $f = f(s,x)$ is a suitable random function and M is a worthy martingale measure, we can construct a new martingale measure $f \bullet M$, where for $A \in \mathcal{A}$,

$$(f \bullet M)([0,t] \times A) = \int_0^t \int_A f(s,x) M(dx\,ds).$$

We can therefore derive the following martingale measures from M and N.

$$\begin{aligned} W_u &= b(u(\cdot,\cdot))^{-1} \bullet M, \\ W_v &= b(v(\cdot,\cdot))^{-1} \bullet N. \end{aligned}$$

Notice that since b is bounded away from zero, we don't have any problems with degeneracy. Applying Theorem 2.5 of [32] to compute the covariance measures gives

$$Q_{W_u}(dx\,dy\,ds) = Q_{W_v}(dx\,dy\,ds) = \delta_{\{x=y\}}\,dx\,dy\,ds$$

so by Proposition 2.10 of [32], W_u and W_v are standard white noises. Also,

$$\begin{aligned}\langle W_u(A), W_v(B)\rangle_t &= \int_0^t \int_{A\cap B} b(u(s,x))b(v(s,x))Q_{M,N}(dx\,dy\,ds) \\ &= \int_0^t \int_A \int_B Q_{W_u,W_v}(dx\,dy\,ds)\end{aligned}$$

where

$$Q_{W_u,W_v}(dx\,dy\,ds) = \alpha(U(s,x))\delta_{\{x=y\}}\,dx\,dy\,ds.$$

To check that u and v solve (1.1) with respect to W_u and W_v respectively, it is sufficient to show that for $\phi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned}\int_0^t \int b(u(s,x))\phi(x)W_u(dx\,ds) &= \int_0^t \int \phi(x)M(dx\,ds) = M^\phi(t), \\ \int_0^t \int b(v(s,x))\phi(x)W_v(dx\,ds) &= \int_0^t \int \phi(x)N(dx\,ds) = N^\phi(t).\end{aligned}$$

The first equality in each line follows trivially from the definition of W_u and W_v. We check the second by approximating ϕ by a step function and taking limits as follows. Fix $\varepsilon > 0$ and choose $\phi_\varepsilon = \sum_{i=1}^n \lambda_i \chi_{A_i}$, where A_i are intervals, such that $\int(\phi - \phi_\varepsilon)^2 < \varepsilon$. Then

$$\mathbb{E}\left[\left(M^\phi(t) - \int_0^t \int \phi(x)\,M(dx\,ds)\right)^2\right] \leq$$
$$2\mathbb{E}\left[\left(M^\phi(t) - \sum_{i=1}^n \lambda_i M_t(A_i)\right)^2\right] + 2\mathbb{E}\left[\int_0^t \int (\phi_\varepsilon(x) - \phi(x))^2\,M(dx\,ds)\right].$$

The second term on the right is bounded by a constant times ε. Now for each $i = 1 \ldots n$, choose a function $\phi_i \in C^\infty(\mathbb{R})$ such that $\mathbb{E}\left[(M^{\phi_i}(t) - M_t(A_i))^2\right]$ and $\int \lambda_i^2(\phi_i - \chi_{A_i})^2$ are both less than ε/n^3. Then the first expectation on the right is bounded by a constant times

$$\begin{aligned}
&\mathbb{E}\left[\left(M^{\phi-\sum\lambda_i\phi_i}(t)\right)^2\right] + \mathbb{E}\left[\left(\sum_{i=1}^n \lambda_i \left(M^{\phi_i}(t) - M_t(A_i)\right)\right)^2\right] \\
&\leq \sup_{s\leq t, y\in\mathbb{R}} \mathbb{E}\left[b(u(s,y))^2\right] \int_0^t \int \left(\phi(x) - \sum_{i=1}^n \lambda_i\phi_i(x)\right)^2 dx\, ds + \\
&\quad n^2 \sum_{i=1}^n \mathbb{E}\left[(M_t(A_i) - M^{\phi_i}(t))^2\right].
\end{aligned}$$

The second term on the right is bounded by a constant times ε. The first term is bounded by a constant times

$$\int \left(\sum_{i=1}^n \lambda_i(\phi_i(x) - \chi_{A_i}(x))\right)^2 dx + \int (\phi_\varepsilon(x) - \phi(x))^2\, dx$$

which is easily seen to also be bounded by a constant multiple of ε. Hence $\int_0^t \int \phi\, dM = M^\phi(t)$ almost surely as required.

Finally, we construct the pair of noises W_1 and W_2. Set $W_1 = W_u$. We would like to construct W_2 such that

$$W_v = \alpha(U(\cdot,\cdot)) \bullet W_1 + \beta(U(\cdot,\cdot)) \bullet W_2 \tag{1.35}$$

so our first thought might be to define

$$W_2 = \beta(U(\cdot,\cdot))^{-1} \bullet (W_v - \alpha(U(\cdot,\cdot)) \bullet W_u)$$

but unfortunately, we have a degeneracy problem where $\beta(U) = 0$. To remedy this, we add an extra white noise $\tilde{W}$ to our space which is independent of everything else, and define

$$W_2 = \frac{\chi_{\{\beta(U(\cdot,\cdot))\neq 0\}}}{\beta(U(\cdot,\cdot))} \bullet (W_v - \alpha(U(\cdot,\cdot)) \bullet W_u) + \chi_{\{\beta(U(\cdot,\cdot))=0\}} \bullet \tilde{W}$$

where we agree that $\chi_{\{\beta(U)\neq 0\}}/\beta(U) = 0$ if $\beta(U) = 0$. Now

$$\begin{aligned}
&Q_{W_2}(dx\,dy\,ds)\\
&\quad= \frac{\chi_{\{\beta(U(s,x))\neq 0\}}}{\beta(U(s,x))^2}\Big(Q_{W_v}(dx\,dy\,ds) - 2\alpha(U(s,x))Q_{W_u,W_v}(dx\,dy\,ds)\\
&\qquad\qquad +\alpha(U(s,x))^2 Q_{W_u}(dx\,dy\,ds)\Big) + \chi_{\{\beta(U(s,x))=0\}} Q_{\tilde{W}}(dx\,dy\,ds)\\
&\quad= \left[\frac{\chi_{\beta(U(s,x))\neq 0}}{\beta(U(x,x))^2}\left(1-\alpha(U(s,x))^2\right) + \chi_{\{\beta(U(s,x))=0\}}\right]\delta_{\{x=y\}}\,dx\,dy\,ds\\
&\quad= \delta_{\{x=y\}}\,dx\,dy\,ds
\end{aligned}$$

so W_2 is a standard white noise. Also,

$$\begin{aligned}
&\langle W_1(A), W_2(B)\rangle_t\\
&\quad= \int_0^t \int_A \int_B \frac{\chi_{\{\beta(U(s,x))\neq 0}}{\beta(U(s,x))}\left(Q_{W_u,W_v}(dx\,dy\,ds) - \alpha(U(s,x))Q_{W_u}(dx\,dy\,ds)\right).
\end{aligned}$$

Since $Q_{W_u,W_v}(dx\,dy\,ds) = \alpha(U(s,x))Q_{W_u}(dx\,dy\,ds)$, this is zero and so W_1 and W_2 are independent. Since $\alpha(U) = 1$ when $\beta(U) = 0$, we can also see that

$$\chi_{\{\beta(U(\cdot,\cdot))=0\}} \bullet (W_v - \alpha(U(\cdot,\cdot)) \bullet W_u) = 0$$

so

$$\beta(U(\cdot,\cdot)) \bullet W_2 = W_v - \alpha(U(\cdot,\cdot)) \bullet W_u$$

and therefore (1.35) is satisfied. □

Proposition 1.5.2 *Suppose $f, g \in C^+_{tem}$, $f \geq g$, and $\theta > 0$ is a fixed parameter. Then there exists a probability space with a pair of continuous, C_{tem}-valued processes $(u(t), v(t))$ with $(u(0), v(0)) = (f, g)$, and a white noise W, such that $u \geq v \geq 0$ almost surely and u and v solve*

$$
\begin{aligned}
\partial_t u &= \Delta u + \theta u - u^2 + \sqrt{2v}\, dW \\
\partial_t v &= \Delta v + \theta v - v^2 + \sqrt{2v}\, dW.
\end{aligned}
$$

The proof of the above proposition is similar to that of the following:

Proposition 1.5.3 *Suppose $f, g \in C^+_{tem}$, $f \geq g$, and $\theta > 0$ is a fixed parameter. Suppose $a : \mathbb{R} \to \mathbb{R}$ is Lipschitz with $a(u) \geq \theta u - u^2$ for all $u \geq 0$. Then there exists a probability space with a pair of continuous, C_{tem}-valued processes $(u(t), v(t))$ with $(u(0), v(0)) = (f, g)$, and a white noise W, such that $u \geq v \geq 0$ almost surely and u and v solve*

$$
\left.
\begin{aligned}
\partial_t u &= \Delta u + a(u) + \sqrt{2u}\, dW \\
\partial_t v &= \Delta v + \theta v - v^2 + \sqrt{2v}\, dW.
\end{aligned}
\right\} \tag{1.36}
$$

Proof Again, we use Lipschitz approximations to the coefficients and check tightness. Set $b_n(u) = \sqrt{2|u| + 1/n} - 1/n$ and $\gamma_n(u) = \theta u - (|u| \wedge n)|u|$. Clearly, $a(u) \geq \gamma_n(u)$ for $|u| \leq n$. Suppose L is the Lipschitz coefficient of a. Then for $|u| \geq n$, $a(u) \geq a(u) - a(0) \geq -L|u|$ and $\gamma_n(u) = \theta u - n|u| \leq |u|(\theta - n)$ so for $n \geq \theta + L$, $a \geq \gamma_n$. Hence for sufficiently large n we can construct solutions $(\bar{u}_n, \bar{v}_n)$ to

$$
\begin{aligned}
\partial_t \bar{u}_n &= \Delta \bar{u}_n + a(\bar{u}_n) + b_n(\bar{u}_n)\, dW \\
\partial_t \bar{v}_n &= \Delta \bar{v}_n + \gamma_n(\bar{v}_n) + b_n(\bar{v}_n)\, dW
\end{aligned}
$$

such that $0 \leq \tilde{v}_n \leq \tilde{u}_n$ almost surely by Proposition 1.2.2 and Theorem 2.3 of Shiga [29]. To check the tightness condition in Proposition 1.9.1, by Lemmas 1.3.4 and 1.3.5 we need to check that for each $p \geq 2$ and $T, \lambda > 0$,

$$\mathbb{E}\left[\left(|a(\tilde{u}_n(t,x))| + |\gamma_n(\tilde{v}_n(t,x))| + |b_n(\tilde{u}_n(t,x))| + |b_n(\tilde{v}_n(t,x))| \right)^p e^{-\lambda|x|} \right]$$

is bounded independently of $n \in \mathbb{N}$, $x \in \mathbb{R}$ and $t \leq T$. Since $|a(u)| + |b_n(u)| + |\gamma(u)|$ is bounded by a constant times $1 + |u|^2$, this reduces to checking

$$\sup_{n \in \mathbb{N}} \sup_{x \in \mathbb{R}, t \leq T} \mathbb{E}\left[\left(|\tilde{u}_n(t,x)| + |\tilde{v}_n(t,x)| \right)^p e^{-\lambda|x|} \right] < \infty$$

for $p \geq 2$ and $T, \lambda > 0$. Note that by Proposition 1.2.2, $0 \leq \tilde{v}_n \leq \bar{\tilde{v}}_n$ almost surely, where $\bar{\tilde{v}}_n$ solves

$$\partial_t \bar{\tilde{v}}_n = \Delta \bar{\tilde{v}}_n + \theta \bar{\tilde{v}}_n + b_n(\bar{\tilde{v}}_n)\, dW$$

so it is sufficient to check moment bounds for $\bar{\tilde{v}}_n$. These can be obtained by Lemma 1.1.3. The case for $\tilde{u}_n$ is similar, except we don't have a $-u^2$ term to worry about.

Proceeding in a similar manner to the previous case, we pass to the limit and construct a pair of processes (u, v) which solve the martingale problem corresponding to (1.36), and martingale measures $M(dx\,ds)$ and $N(dx\,ds)$ such that

$$\begin{aligned}
Q_M(dx\,dy\,ds) &= 2u(s,x)\delta_{\{x=y\}}\, dx\,dy\,ds \\
Q_N(dx\,dy\,ds) &= 2v(s,x)\delta_{\{x=y\}}\, dx\,dy\,ds \\
Q_{M,N}(dx\,dy\,ds) &= \sqrt{4u(s,x)v(s,x)}\,\delta_{\{x=y\}}\, dx\,dy\,ds.
\end{aligned}$$

Since u and v are constructed as almost sure limits of u_n and v_n, where $u_n \geq v_n \geq 0$, it is clear that $u \geq v \geq 0$. We would like to define our white noises by

$$\begin{aligned} W_u &= (2u(\cdot,\cdot))^{-1/2} \bullet M \\ W_v &= (2v(\cdot,\cdot))^{-1/2} \bullet N \end{aligned}$$

but again we have a degeneracy problem when u or v are zero. As before, we add a white noise $\tilde{W}$ which is independent of everything else in our space, and define

$$\begin{aligned} W_u &= \frac{\chi_{\{u(\cdot,\cdot)\neq 0\}}}{\sqrt{2u(\cdot,\cdot)}} \bullet M + \frac{\chi_{\{u(\cdot,\cdot)=0, v(\cdot,\cdot)\neq 0\}}}{\sqrt{2v(\cdot,\cdot)}} \bullet N + \chi_{\{u(\cdot,\cdot)=v(\cdot,\cdot)=0\}} \bullet \tilde{W} \\ W_v &= \frac{\chi_{\{v(\cdot,\cdot)\neq 0\}}}{\sqrt{2v(\cdot,\cdot)}} \bullet N + \frac{\chi_{\{v(\cdot,\cdot)=0, u(\cdot,\cdot)\neq 0\}}}{\sqrt{2u(\cdot,\cdot)}} \bullet M + \chi_{\{u(\cdot,\cdot)=v(\cdot,\cdot)=0\}} \bullet \tilde{W}. \end{aligned}$$

Computing the covariance measures of W_u and W_v, we see that they are standard white noises. By definition, for all $\phi \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned} \int_0^t \int \sqrt{2u(s,x)}\,\phi(x)\,dW_u &= \int_0^t \int \phi(x)\,M(dx\,ds) \\ \int_0^t \int \sqrt{2v(s,x)}\,\phi(x)\,dW_v &= \int_0^t \int \phi(x)\,N(dx\,ds) \end{aligned}$$

which are equal to $M^\phi(t)$ and $N^\phi(t)$ respectively as before. Hence u and v solve the required SPDEs. Furthermore, for bounded borel subsets A, B of $\mathbb{R}$,

$$\langle W_u(A), W_v(B)\rangle_t = |A \cap B|t$$

so it follows that $\langle W_u(A) - W_v(B)\rangle_t = 0$ and therefore W_u and W_v are actually the same noise. □

As a corollary of the construction of a solution of an SPDE from its associated martingale problem, we can show that uniqueness in law of a solution gives uniqueness in law of its associated martingale problem.

Corollary 1.5.4 *Choose $f \in C_{tem}$. Suppose that either*

1. *a and b are Lipschitz and $f \in C_{tem}$, or*

2. *$a(u) = \theta u - u^2$ for some $\theta > 0$, $b(u) = \sqrt{2u}$ and $f \in C^+_{tem}$.*

Suppose that u is a C_{tem}-valued continuous process such that

$$\left.\begin{array}{rcl} M^\phi(t) & = & (u(t) - u(0), \phi) - \int_0^t [(u(s), \Delta\phi) + (a(u(s,\cdot)), \phi)]\, ds \\ & & M^\phi(t)^2 - \int_0^t \int b(u(s,x))^2 \phi(x)^2\, dx\, ds \end{array}\right\} \quad (1.37)$$

are both martingales for each $\phi \in C_c^\infty(\mathbb{R})$. Then the law of u is also uniquely determined.

Proof Given a solution u of the martingale problem (1.37), we can apply the procedure described above to construct a white noise W with respect to which u is a C_{tem}-valued continuous solution of (1.1). Since uniqueness in law of C_{tem}-valued continuous solutions holds in both cases (see Tribe [31] for case 2), the law of u is uniquely determined. □

1.6 Solutions with random initial conditions

So far, we have only constructed solutions of (1.1) for deterministic initial conditions. We now construct, given a suitable probability distribution μ on C_{tem}, a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ containing a C_{tem}-valued continuous process u and a white noise W such that (1.2) is satisfied $\mathbb{P}$-almost surely and $u(0)$ has distribution μ.

We work with the canonical setup, i.e. set $\Omega = C(\mathbb{R}^+ \to C_{tem})$, $\mathcal{F} = \mathcal{B}(\Omega)$, the Borel sets under the metric on Ω defined earlier, and set

$$\mathcal{F}_t = \sigma\left(|_{[0,t]}^{-1}(A) \ : \ A \in \mathcal{B}(C([0,t] \to C_{tem}))\right)$$

where $|_{[0,t]} : C(\mathbb{R}^+ \to C_{tem}) \to C([0,t] \to C_{tem})$ is the restriction map. Define the canonical random variable u by $u(t,x,\omega) = \omega(t)(x)$. If a and b are Lipschitz, for each $f \in C_{tem}$ we can construct a probability measure $\mathbb{P}_f$ induced on Ω by a solution of (1.1) with initial condition f. For the non-Lipschitz cases we have treated, we can construct $\mathbb{P}_f$ as a weak limit of a sequence $\mathbb{P}_f^n$ of distributions of Lipschitz approximations, although here, $\mathbb{P}_f$ may not be uniquely defined. We then set

$$\mathbb{P}_\mu = \int \mathbb{P}_f \, \mu(df).$$

For this definition to make sense, we need to check that for all bounded Borel-measurable $H : \Omega \to \mathbb{R}$, the function $C_{tem} \to \mathbb{R}$ given by $f \mapsto \int H(\omega)\, \mathbb{P}_f(d\omega)$ is measurable, although it is sufficient to check this for H continuous by a monotone class argument. In the Lipschitz case, and if H is continuous, it follows from Corollary 1.4.3 that it is continuous. Measurability follows for the more general case, since

$$\int H(\omega)\, \mathbb{P}_f(d\omega) = \lim_{n\to\infty} \int H(\omega)\, \mathbb{P}_f^n(d\omega)$$

and we can approximate a Borel function by a sequence of continuous ones.

We can show u is a solution of (1.1) by checking that the relevant martingale problem is satisfied in a similar manner to Section 1.5. For each $f \in C_{tem}$ and each $\phi \in C_c^\infty(\mathbb{R})$, (1.37) are martingales with respect to $\mathbb{P}_f$. To show that they are also $\mathbb{P}_\mu$-martingales, it is sufficient to check that they are integrable with respect to $\mathbb{P}_\mu$. This is true if, for all $N, T > 0$,

$$\sup_{y\in[-N,N],s\leq T} \mathbb{E}_\mu \left[a(u(s,y))^2 + b(u(s,y))^2 + u(s,y)^2\right] < \infty.$$

In the case where (1.3) is satisfied, by Lemma 1.1.3 this is given by

$$\int \left(\sup_{y\in\mathbb{R}} f(y)^2 e^{-\lambda|y|}\right) \mu(df) < \infty \text{ for each } \lambda > 0. \tag{1.38}$$

The only situation we have considered where (1.3) does not hold is when $a(u) = \theta u - u^2$, but here we see that (1.38) is still sufficient by using the comparison result to compare to the case $a(u) = \theta u$. We can now follow the procedure given in Section 1.5 earlier to construct the required white noise.

We will also need the following uniqueness result.

Lemma 1.6.1 *Suppose a and b are such that for each $f \in C_{tem}$, there is a unique probability measure $\mathbb{P}_f$ on Ω with respect to which (1.37) are martingales for each $\phi \in C_c^\infty(\mathbb{R})$ and $\omega(0) = f$ almost surely. Suppose $\mathbb{P}$ is a probability measure on Ω such that for each $\phi \in C_c^\infty(\mathbb{R})$, (1.37) are $\mathbb{P}$-martingales, and $\mathbb{P}(u(0) \in \cdot) = \mu$ for some distribution μ on C_{tem}. Then*

$$\mathbb{P} = \int \mathbb{P}_f \, \mu(df).$$

Proof To prove this result, we need the concept of *regular conditional probabilities*. By Theorem II 89.1 of Rogers and Williams [27], there exists a function $(\mathbb{P}|\mathcal{F}_0) : \mathcal{F} \times \Omega \to [0,1]$ such that

1. for each $F \in \mathcal{F}$, $\omega \mapsto (\mathbb{P}|\mathcal{F}_0)(F,\omega)$ is a version of $\mathbb{P}(F|\mathcal{F}_0)$,

2. for each $\omega \in \Omega$, the map $F \mapsto (\mathbb{P}|\mathcal{F}_0)(F,\omega)$ is a probability measure on $\mathcal{F}$,

3. for $\mathbb{P}$-almost all $\omega \in \Omega$, $(\mathbb{P}|\mathcal{F}_0)(G,\omega) = \chi_G(\omega)$ for all $G \in \mathcal{F}_0$.

We first show that for $\mathbb{P}$-almost all $\omega \in \Omega$, $(\mathbb{P}|\mathcal{F}_0)(\cdot,\omega) = \mathbb{P}_{\omega(0)}$. We do this by showing that the left-hand side solves the martingale problem and applying uniqueness. By 3, $(\mathbb{P}|\mathcal{F}_0)(\cdot,\omega)$ has initial distribution $\omega(0)$ for $\mathbb{P}$-almost all ω. Suppose M is a $\mathbb{P}$-martingale. Choose $F \in \mathcal{F}_s$ and $G \in \mathcal{F}_0$. Then

$$\begin{aligned} 0 &= \int (M_t(\omega) - M_s(\omega))\, \chi_F(\omega)\chi_G(\omega)\, \mathbb{P}(d\omega) \\ &= \int \chi_G(\omega') \left(\int (M_t(\omega) - M_s(\omega))\, \chi_F(\omega)\, (\mathbb{P}|\mathcal{F}_0)(d\omega,\omega') \right) \mathbb{P}(d\omega'). \end{aligned}$$

Since the inner integral is $\mathcal{F}_0$-measurable, we see that for $\mathbb{P}$-almost all ω',

$$\int (M_t(\omega) - M_s(\omega))\, \chi_F(\omega)(\mathbb{P}|\mathcal{F}_0)(d\omega,\omega') = 0. \tag{1.39}$$

Because $\mathcal{F}_s$ is countably generated, (1.39) holds for all $F \in \mathcal{F}_s$, and hence M is also a $(\mathbb{P}|\mathcal{F}_0)(\cdot,\omega')$-martingale, for $\mathbb{P}$-almost all ω'. It follows by uniqueness that $(\mathbb{P}|\mathcal{F}_0)(\cdot,\omega) = \mathbb{P}_{\omega(0)}$ for $\mathbb{P}$-almost all ω. Now choose $F \in \mathcal{F}$. Then

$$\mathbb{P}(F) = \int (\mathbb{P}|\mathcal{F}_0)(F,\omega)\, \mathbb{P}(d\omega) = \int \mathbb{P}_{\omega(0)}(F)\, \mathbb{P}(d\omega) = \int \mathbb{P}_f(F)\, \mu(df)$$

as required. □

1.7 The Markov property

Given coefficients a and b of (1.1), for each $f \in C_{tem}$ and $t > 0$, we let $\Theta_t f$ be the probability distribution on C_{tem} given by

$$\Theta_t f(A) := \mathbb{P}_f(u(t) \in A).$$

We assume that a and b satisfy the hypotheses of Lemma 1.6.1, and so this is well defined. More generally, given a distribution μ on C_{tem}, we set

$$\Theta_t\mu(A) = \mathbb{P}_\mu(u(t) \in A) = \int \Theta_t f(A)\,\mu(df).$$

We say a distribution μ is *stationary* with respect to a and b if, for all $t > 0$, $\Theta_t\mu = \mu$. The main aim of the thesis is to prove existence and uniqueness of stationary distributions.

In order to be able to prove existence of stationary distributions, we need to show that for $s, t > 0$, $\Theta_t(\Theta_s\mu) = \Theta_{t+s}\mu$ (i.e. solutions satisfy a Markov condition). To prove this, we follow the method of Theorem 4.2 on page 184 of Ethier and Kurtz [8] as follows.

Lemma 1.7.1 *Suppose a and b satisfy the hypotheses of Lemma 1.6.1, and for all $g \in C_{tem}$, $s \geq 0$ and $\lambda > 0$,*

$$\mathbb{E}_g\left[\sup_{y\in\mathbb{R}} u(s,y)^2 e^{-\lambda|y|}\right] < \infty. \tag{1.40}$$

Then, for all $s, t \geq 0$,

$$\Theta_t(\Theta_s\mu) = \Theta_{t+s}\mu. \tag{1.41}$$

Proof First note that it is sufficient to check $\Theta_t(\Theta_s g) = \Theta_{t+s}g$ for each $g \in C_{tem}$, since the more general case will follow by integrating over μ.

The idea is to define two probability measures $\mathbb{P}_1$ and $\mathbb{P}_2$, each with initial distribution $\Theta_s g$, and with respect to which (1.37) are martingales for each $\phi \in C_c^\infty(\mathbb{R})$, then use uniqueness to assert that $\mathbb{P}_1$ and $\mathbb{P}_2$ must be the same. Fix $s \geq 0$

and for $B \in \mathcal{F}$, set

$$\begin{aligned} \mathbb{P}_1(B) &= \int \mathbb{P}_f(B)(\Theta_s g)(df) \\ \mathbb{P}_2(B) &= \mathbb{P}_g(u(s+\cdot) \in B). \end{aligned}$$

$\mathbb{P}_f$ solves the martingale problem (1.37) for each $f \in C_{tem}$. By (1.40), $\Theta_s g$ satisfies (1.38), so $\mathbb{P}_1$ solves the martingale problem. To see that $\mathbb{P}_2$ is also a solution to the martingale problem, we note that

$$\begin{aligned} (u(t+s)-u(s),\phi) &- \int_0^t ((u(r+s),\Delta\phi) + (a(u(r+s,\cdot)),\phi))\,dr \\ &= M^\phi(t+s) - M^\phi(s) \end{aligned}$$

which is a martingale in t with respect to $\mathbb{P}_g$. Also,

$$\begin{aligned} &\left((u(t+s)-u(s),\phi) - \int_0^t ((u(r+s),\Delta) + (a(u(r+s,\cdot)),\phi))\,dr\right)^2 \\ &- \int_0^t \int b(u(r+s,x))^2\,dx\,dr \\ &= M^\phi(t+s)^2 - \int_0^{t+s} \int b(u(r,x))^2\phi(x)^2\,dx\,dr \\ &+ M^\phi(s)^2 + \int_0^s \int b(u(r,x))^2\phi(x)^2\,dx\,dr - 2M^\phi(t+s)M^\phi(s) \end{aligned}$$

which is also a martingale with respect to $\mathbb{P}_g$. Hence $u(s+\cdot)$ is a solution of the martingale problem with respect to $\mathbb{P}_g$ and therefore $\mathbb{P}_2$ is also a solution of the martingale problem.

Since $\mathbb{P}_1$ and $\mathbb{P}_2$ both have initial distribution $\Theta_s g$, by Lemma 1.6.1, $\mathbb{P}_1 = \mathbb{P}_2$ and in particular, $\mathbb{P}_1(u(t) \in B) = \mathbb{P}_2(u(t) \in B)$ for all $B \in \mathcal{B}(C_{tem})$, i.e.

$$\int \mathbb{P}_f(u(t) \in B)(\Theta_s g)(df) = \mathbb{P}_g(u(s+t) \in B).$$

This gives (1.41) as required. □

Applying the above lemma and Corollary 1.5.4, we see that (1.41) holds for the two cases considered in Corollary 1.5.4.

1.8 Existence of a stationary distribution

In this section we prove that, provided that the moments don't blow up, (1.1) has a stationary distribution. We use the standard method due to Krylov and Bogoliubov, which is given in the proof of Theorem 3.1.1 of Da Prato and Zabczyk [3].

Proposition 1.8.1 *Suppose a and b are Lipschitz and let u be the solution of (1.1) with initial condition $g \in C_{tem}$. Suppose we have, for all $p \in \mathbb{N}$ and $\lambda > 0$,*

$$\sup_{s \geq 0, y \in \mathbb{R}} \mathbb{E}\left[|u(s,y)|^p e^{-\lambda|y|}\right] < \infty. \tag{1.42}$$

For each $t > 0$, set $\nu_t = t^{-1}\int_0^t (\Theta_s g)\, ds$, where $\Theta_s g$ is the distribution of $u(s)$ on C_{tem}. Then ν_t converges weakly along a subsequence t_n to a stationary distribution ν.

Proof By Corollary 1.4.3, if $H : C(\mathbb{R}^+ \to C_{tem}) \to \mathbb{R}$ is continuous and bounded, $f \mapsto \int H(\omega)\, \mathbb{P}_f(d\omega)$ is continuous. In particular, if $H : C_{tem} \to \mathbb{R}$ is continuous and bounded,

$$f \mapsto \int H(\omega)(\Theta_t f)(d\omega) \tag{1.43}$$

is continuous for each $t \geq 0$.

By Lemma 1.3.7 and (1.42), $\{\Theta_t g\}_{t\geq\delta}$ is tight for each $\delta > 0$. To see $\{\nu_t\}_{t\geq 1}$ is tight, for each $\varepsilon > 0$ choose $\delta \in (0, \varepsilon/2)$ then choose a compact set $K \in C(\mathbb{R}_+ \to C_{tem})$ such that $\inf_{s\geq\delta}(\Theta_s g)(K) \geq 1 - \varepsilon/2$. Then for $t \geq 1$,

$$\nu_t(K) = \frac{1}{t}\int_0^t (\Theta_s g)(K)\, ds \geq \left(\frac{t-\delta}{t}\right) \inf_{s\geq\delta}(\Theta_s g)(K) \geq 1 - \varepsilon.$$

Hence ν_{t_n} converges weakly to a distribution ν along some subsequence t_n.

Let $H : C_{tem} \to \mathbb{R}$ be continuous and bounded. Then by (1.43) and the definition of weak convergence,

$$\lim_{n\to\infty} \int\int H(\omega)(\Theta_T f)(d\omega)\, \nu_{t_n}(df) = \int\int H(\omega)(\Theta_T f)(d\omega)\, \nu(df)$$

which equals $\int H(d\omega)(\Theta_T \nu)(d\omega)$. Now

$$\begin{aligned}
&\int\int H(\omega)(\Theta_T f)(d\omega)\, \nu_{t_n}(df) \\
&\quad = \frac{1}{t_n}\int_0^{t_n}\int\int H(\omega)(\Theta_T f)(d\omega)\, (\Theta_s g)(df)\, ds \\
&\quad = \frac{1}{t_n}\int_0^{t_n}\int H(\omega)(\Theta_T(\Theta_s g))(d\omega)\, ds \\
&\quad = \frac{1}{t_n}\int_0^{t_n}\int H(\omega)(\Theta_{T+s} g)(d\omega)\, ds \\
&\quad = \frac{1}{t_n}\int_T^{t_n+T}\int H(\omega)(\Theta_s g)(d\omega)\, ds \\
&\quad = \frac{1}{t_n}\left(\int_0^{t_n} + \int_{t_n}^{t_n+T} - \int_0^T\right) H(\omega)(\Theta_s g)(d\omega)\, ds
\end{aligned}$$

which converges to $\int H(\omega)\, \nu(d\omega)$. So for all continuous bounded H we have

$$\int H(\omega)(\Theta_T \nu)(d\omega) = \int H(\omega)\, \nu(d\omega)$$

and therefore $\Theta_T \nu = \nu$ as required. $\square$

1.9 Tightness conditions

This result is similar to the one stated without proof as Lemma 6.3(ii) of Shiga [29]. Since it was a little unclear which norm he was using on $C([0,1] \to C_{tem})$, and he omits the $e^{\lambda|x|}$ term from his statement, we supply a proof.

Proposition 1.9.1 *Suppose $\{X_n(s)\}_{n\in\mathbb{N}}$ is a family of continuous, C_{tem}-valued processes such that for all $T > 0$ and $\lambda > 0$ there exist $p > 1$, $C < \infty$ and $\gamma \geq 8$ such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}[|X_n(t,y_1) - X_n(s,y_2)|^p] \leq C(|t-s|^\gamma + |y_1 - y_2|^\gamma)e^{\lambda|x|}$$

for all $x \in \mathbb{R}$, $y_1, y_2 \in B_1(x)$ and $0 \leq s, t \leq T$ with $|s-t| \leq 1$, and

$$\sup_{n\in\mathbb{N}} \mathbb{E}[|X_n(0,x)|^p] \leq Ce^{\lambda|x|}$$

for all $x \in \mathbb{R}$. Then $\{X_n\}$ is tight on $C(\mathbb{R}^+ \to C_{tem})$ with respect to the metric on this space given earlier.

We also need a tightness result for variables on C_{tem} — the proof of this is omitted since it is similar to that of Proposition 1.9.1 but without time increments.

Proposition 1.9.2 *Suppose $\{X(s)\}_{s\geq 0}$ is a family of C_{tem}-valued random variables such that for all $\lambda > 0$ there exist $p > 1$, $C < \infty$ and $\gamma \geq 8$ such that*

$$\sup_{s\geq 0} \mathbb{E}[|X(s,y_1) - X_n(s,y_2)|^p] \leq C|y_1 - y_2|^\gamma e^{\lambda|x|}$$

for all $x \in \mathbb{R}$ and $y_1, y_2 \in B_1(x)$, and

$$\sup_{s\geq 0} \mathbb{E}[|X(s,x)|^p] \leq Ce^{\lambda|x|}$$

for all $x \in \mathbb{R}$. Then $\{X(s)\}$ is tight on C_{tem}.

We first prove a lemma characterising compact subsets of $C(\mathbb{R}^+ \to C_{tem})$.

Lemma 1.9.3 *Suppose $K \subset C(\mathbb{R}^+ \to C_{tem})$ is closed and*

1. $\sup_{f\in K} \sup_{s\in[0,T]} \|f\|_\lambda < \infty$ *for each* $\lambda > 0$, $T > 0$

2. $K_N := \{f|_{[0,N]\times[-N,N]} : f \in K\}$ *is equicontinuous in* $C([0,N] \times [-N,N])$ *for each* $N \in \mathbb{N}$.

Then K is compact.

Proof We take a sequence $\{f_n\}$ in K. By the Arzela-Ascoli Theorem, for each $N \in \mathbb{N}$ we may take a subsequence $\{f_{n_r^N}\}$ such that $f_{n_r^N}|_{[0,N]\times[-N,N]}$ converges uniformly. By a diagonalisation argument, we obtain a subsequence $\{f_{n_r}\}$ and $f \in C(\mathbb{R}^+ \times \mathbb{R})$ such that $f_{n_r}|_{[0,N]\times[-N,N]}$ converges uniformly to $f|_{[0,N]\times[-N,N]}$ for each $N \in \mathbb{N}$. It remains to show that f is in $C(\mathbb{R}^+ \to C_{tem})$ and that f_{n_r} converges to f in the $C(\mathbb{R}^+ \to C_{tem})$ metric.

$$\sup_{s\leq T} \sup_{x\in[-N,N]} e^{-\lambda|x|}|f(s,x)| \leq \sup_{s\leq T} \sup_{x\in[-N,N]} |f(s,x) - f_{n_r}(s,x)| + \sup_{s\leq T} \|f_{n_r}(s)\|_\lambda.$$

By letting $r \to \infty$, we see that for each $N \in \mathbb{N}$,

$$\sup_{s\leq T} \sup_{x\in[-N,N]} e^{-\lambda|x|}|f(s,x)| \leq \sup_{g\in K} \sup_{s\leq T} \|g(s)\|_\lambda < \infty$$

and hence $\sup_{s\leq T} \|f(s)\|_\lambda$ is bounded for each $T > 0$ and $\lambda > 0$. In particular, f is C_{tem}-valued. To show f is a continuous function $\mathbb{R}^+ \to C_{tem}$, take $t \leq T$ and $\varepsilon > 0$. Then

$$\begin{aligned} d_{tem}(f(s), f(t)) &= \sum_{\lambda\in 1/\mathbb{N}} 2^{-1/\lambda} (\|f(s) - f(t)\|_\lambda \wedge 1) \\ &\leq \|f(s) - f(t)\|_\lambda + \frac{\varepsilon}{3} \end{aligned}$$

for small enough λ. Now, if $s \le t+1$,

$$\begin{aligned}
&\|f(s) - f(t)\|_\lambda \\
&\quad\le \sup_{x\in[-N,N]} |f(s,x) - f(t,x)| + e^{-\lambda N/2}\|f(s) - f(t)\|_{\lambda/2} \\
&\quad\le \sup_{x\in[-N,N]} |f(s,x) - f(t,x)| + \left(2 \sup_{r\le T+1} \sup_{g\in K} \|g(r)\|_{\lambda/2}\right) e^{-\lambda N/2} \\
&\quad\le \sup_{x\in[-N,N]} |f(s,x) - f(t,x)| + \frac{\lambda\varepsilon}{3}
\end{aligned}$$

for sufficiently large N, so we see that $d_{tem}(f(s), f(t)) < \varepsilon$ for s sufficiently close to t. Also,

$$\begin{aligned}
d(f_{n_r}, f) &= \sum_{T=1}^{\infty} 2^{-T} \sup_{s\le T} d_{tem}(f_{n_r}(s), f(s)) \\
&\le \sup_{s\le T} d_{tem}(f_{n_r}(s), f(s)) + \frac{\varepsilon}{4} \\
&\le \frac{1}{\lambda} \sup_{s\le T} \|f_{n_r}(s) - f(s)\|_\lambda + \frac{\varepsilon}{2}
\end{aligned}$$

for small enough λ and large enough T. Proceeding in a similar manner to the continuity proof, we see that f_{n_r} converges to f. Since K is closed, $f \in K$ and hence K is compact. □

Proof (of 1.9.1) We construct a subset of $C(\mathbb{R}^+ \to C_{tem})$ which satisfies the hypotheses of the previous lemma and contains each X_n with high probability.

Fix $\lambda > 0$, $T > 0$ and let $p = p(\lambda, T)$, $\gamma = \gamma(\lambda, T)$ be as given in the hypotheses. We let C be a constant depending only on λ and T whose value may change from line to line. For $m \in \mathbb{N}$,

$$\mathbb{E}[|X_n(m,x)|^p] \le C\left(\mathbb{E}[|X_n(0,x)|^p] + \sum_{r=1}^{m} \mathbb{E}[|X_n(r-1,x) - X_n(r,x)|^p]\right)$$

which is bounded by $Ce^{\lambda|x|}$ for $m \leq T$ by the hypotheses. Now

$$\mathbb{E}\left[\sup_{y\in B_1(x),s\in[m,m+1]}|X_n(s,y)|^p\right] \leq C\mathbb{E}[|X_n(m,x)|^p] + C\mathbb{E}\left[\sup_{y\in B_1(x),s\in[m,m+1]}|X_n(s,y)-X_n(m,x)|^p\right]$$

which is again bounded by $Ce^{\lambda|x|}$ by Lemma 1.3.2. Splitting the time interval $[0,T]$ into unit intervals gives

$$\mathbb{E}\left[\sup_{s\leq T,y\in B_1(x)}|X_n(s,y)|^p\right] \leq Ce^{\lambda|x|}.$$

Now for each $A > 0$,

$$\begin{aligned}
\mathbb{P}\left(\sup_{s\leq T}\|X_n(s)\|_\lambda \geq A\right) &\leq \sum_{r\in\mathbb{Z}}\mathbb{P}\left(\sup_{s\leq T,y\in B_1(r)}|X_n(s,y)|^p \geq A^p e^{\lambda p(|r|-1)}\right)\\
&\leq \frac{e^{\lambda p}}{A^p}\sum_{r\in\mathbb{Z}}\mathbb{E}\left[\sup_{s\leq T,y\in B_1(r)}|X_n(s,y)|^p\right]e^{-\lambda p|r|}\\
&\leq \frac{C}{A^p}\sum_{r\in\mathbb{Z}}e^{\lambda(1-p)|r|}.
\end{aligned}$$

The sum converges since $p > 1$, so for each $\lambda > 0, T > 0$,

$$\sup_{n\in\mathbb{N}}\mathbb{P}\left(\sup_{s\leq T}\|X_n(s)\|_\lambda \geq A\right)$$

can be made arbitrarily small by choosing a suitably large A. Let $\lambda_r = 1/r$ and $T_r = r$, and given $\varepsilon > 0$, choose A_r such that

$$\sup_{n\in I}\mathbb{P}\left(\sup_{s\leq T_r}\|X_n(s)\|_{\lambda_r} \geq A_r\right) < 2^{-r}\varepsilon.$$

Set

$$M_\varepsilon^1 = \left\{f \in C(\mathbb{R}^+ \to C_{tem}) : \sup_{s\leq T_r}\|f(s)\|_{\lambda_r} \leq A_r \text{ for each } r \in \mathbb{N}\right\}.$$

Then M_ε^1 is closed, $\inf_{n\in\mathbb{N}} \mathbb{P}(X_n \in M_\varepsilon) > 1 - \varepsilon$ and M_ε^1 satisfies the first part of the previous lemma.

We now construct a suitable set which is equicontinuous on compacts. Fix $N \in \mathbb{N}$. For $\rho > 0$ and $0 < \delta \leq 1$, we let A be the event that $|X_n(s,x) - X_n(s,y)| < \rho$ for all $x, y \in [-N, N]$, $s, t \leq N$ such that $|x-y| + |s-t| < \delta$. If this event is false then there exists some $x_0 \in \{-N, \ldots, N\}$ and $t_0 \in \{0, \ldots, N\}$ such that for any $\gamma, p > 0$,

$$\sup_{x,y\in B_1(x_0), s,t\in B_1(t_0)} \frac{|X_n(s,x) - X_n(t,y)|^p}{(|x-y| + |s-t|)^{\gamma/2}} > \frac{\rho^p}{\delta^{\gamma/2}}.$$

Applying Markov's inequality, we see that

$$\mathbb{P}(A^c) \leq \frac{\delta^{\gamma/2}}{\rho^p} \sum_{t_0=0}^{N} \sum_{x_0=-N}^{N} \mathbb{E}\left[\sup_{x,y\in B_1(x_0)} \sup_{s,t\in B_1(t_0)} \frac{|X_n(s,x) - X_n(t,y)|^p}{(|x-y| + |t-s|)^{\gamma/2}}\right].$$

By hypotheses and Lemma 1.3.2, the expectations are bounded for some choice of p and γ. So for each $N \in \mathbb{N}$ there exist constants $C, p, \gamma > 0$ such that for all $\rho > 0$, $\delta \in (0, 1]$

$$\mathbb{P}\Bigg(|X_n(s,x) - X_n(s,y)| \geq \rho \text{ for some } |x-y| + |s-t| < \delta,$$
$$(s,x), (t,y) \in [0,N] \times [-N,N] \Bigg) \leq C\frac{\delta^{\gamma/2}}{\rho^p}.$$

Given a positive sequence $\{\delta_r\}_{r\in\mathbb{N}}$, let $K_{\{\delta_r\}}^N$ be the set

$$\Bigg\{ f \in C(\mathbb{R}^+ \to C_{tem}) : \text{for all } r \in \mathbb{N},\ |f(s,x) - f(t,y)| \leq 1/r$$
$$\text{whenever } (s,x), (t,y) \in [0,N] \times [-N,N] \text{ and } |s-t| + |x-y| < \delta_r \Bigg\}.$$

Then for any choice of $\{\delta_r\}$, $K^N_{\{\delta_r\}}$ is closed, and equicontinuous on $[0, N] \times [-N, N]$. Choose $\varepsilon > 0$. For each $r \in \mathbb{N}$ we choose δ_r such that

$$\mathbb{P}\Bigg(|X_n(s,x) - X_n(s,y)| \geq 1/r \text{ for some } |x-y| + |s-t| < \delta_r, \\ (s,x),(t,y) \in [0,N] \times [-N,N] \Bigg) < 2^{-r}\varepsilon$$

for all $n \in \mathbb{N}$. Then $\mathbb{P}\left(X_n \in K^N_{\{\delta_r\}}\right) > 1 - \varepsilon$. Hence, given $N \in \mathbb{N}$ and $\varepsilon > 0$ we can construct a closed set K^N_ε which is equicontinuous on $[0, N] \times [-N, N]$ and such that

$$\sup_{n \in \mathbb{N}} \mathbb{P}\left(X_n|_{[0,N]\times[-N,N]} \notin K^N_\varepsilon\right) < \varepsilon.$$

Set $M^2_\varepsilon = \bigcap_{N \in \mathbb{N}} K^N_{\varepsilon 2^{-N}}$. Then $\inf_{n \in \mathbb{N}} \mathbb{P}(X_n \in M^2_\varepsilon) > 1 - \varepsilon$ and M^2_ε is equicontinuous on compacts as required. We then take $M_\varepsilon = M^1_{\varepsilon/2} \cap M^2_{\varepsilon/2}$ to be our compact subset of $C(\mathbb{R}^+ \to C_{tem})$. $\square$

Chapter 2

Uniqueness of a stationary distribution via coupling methods

2.1 Introduction

The aim of this chapter is to use a coupling technique to show that the long-term behaviour of solutions of a certain class of SPDEs is independent of the initial condition. We consider C_{tem}-valued continuous solutions in one space dimension of SPDEs of the form

$$\partial_t u = \Delta u + a(u) + b(u)\, dW, \tag{2.1}$$

where $a, b : \mathbb{R} \to \mathbb{R}$ are such that there exists a constant $L \geq 1$ such that

1. $|a(u) - a(v)| + |b(u) - b(v)| \leq L|u - v|$ for all $u, v \in \mathbb{R}$,

2. a is nonincreasing,

3. b is bounded in $[1/L, L]$ (so the noise term is nondegenerate),

4. if $f \in C(\mathbb{R})$ is such that for some $\gamma \in (0, 1/3)$ and some $K > 0$,

$$\sup_{y \in \mathbb{R}} \left(\frac{|f(y)|}{1 + |y|^{\gamma}} \right) \leq K, \tag{2.2}$$

and u is a C_{tem}-valued continuous solution of (2.1) with initial condition f, then

$$\sup_{t \geq 0, x \in \mathbb{R}} \mathbb{E} \left[\frac{|u(t,x)|^p}{1 + |x|^{p\gamma}} \right] < \infty \text{ for each } p \geq 2. \tag{2.3}$$

We make condition 2 to ensure that the difference equation we consider later doesn't have a growth term in it, and we only consider initial conditions growing slower than $|x|^{1/3}$ in condition 4 because we need third moments to grow slower than linearly in the proof of Lemma 2.4.4. It will be seen later (Lemma 2.2.1) that condition 4 is satisfied if $a(u) = \bar{a}(u) - \theta u$ for some $\theta > 0$ and $\bar{a}$ bounded. Conditions 1–4 are enough to ensure existence of a stationary distribution via the method we described in Section 1.8, so the rest of the chapter will be concerned with proving uniqueness.

The result is motivated by a paper of C. Mueller [16], but he only considers the case where solutions are defined on S^1, whereas we consider solutions defined on the whole of $\mathbb{R}$. This means that some of his methods, for example considering the total mass of the difference $u - v$ of two solutions, become unavailable.

The general idea of the coupling method is as follows. Suppose we have two solutions of (2.1) with initial distributions μ and ν respectively. For each $t \geq 0$, let μ_t be the distribution of a solution started from μ. Note that this family of distributions is uniquely determined, since a and b are Lipschitz. Let $\{\nu_t\}$ be the similar family of distributions derived from a solution started from ν. Suppose we can construct a single probability space with two solutions u and v started from μ and ν respectively. Suppose A is a measurable subset of C_{tem} depending only on a bounded interval $(-R, R)$. Then

$$\begin{aligned} |\mu_t(A) - \nu_t(A)| = \Big| & \mathbb{P}\left(u(t) \in A \text{ and } u(t)|_{(-R,R)} = v(t)|_{(-R,R)}\right) \\ + & \mathbb{P}\left(u(t) \in A \text{ and } u(t)|_{(-R,R)} \neq v(t)|_{(-R,R)}\right) \\ - & \mathbb{P}\left(v(t) \in A \text{ and } u(t)|_{(-R,R)} = v(t)|_{(-R,R)}\right) \\ - & \mathbb{P}\left(v(t) \in A \text{ and } u(t)|_{(-R,R)} \neq v(t)|_{(-R,R)}\right) \Big| . \end{aligned} \tag{2.4}$$

Since A depends only on $(-R, R)$, the first and third terms on the right cancel and

$$|\mu_t(A) - \nu_t(A)| \leq \mathbb{P}\left(u(t)|_{(-R,R)} \neq v(t)|_{(-R,R)}\right).$$

So if we can construct u and v such that $u(t)$ and $v(t)$ are very likely to coincide on a finite interval as t gets large, we can conclude that the difference between μ_t and ν_t gets small.

We show that for each fixed interval $(-R, R)$ and $\varepsilon > 0$, we can construct a coupling such that, for large time, u and v coincide on $(-R, R)$ with probability at least $1-\varepsilon$. Unlike in Mueller's paper, we will need a different coupling mechanism as ε decreases.

We do this by considering the difference process $U = u - v$ and showing that U behaves in a similar manner to $\partial_t U = \Delta U + \sqrt{U}\, dW$, the density of super-Brownian motion, which is known to exhibit local extinction. Clearly, this method only works if $U \geq 0$ (in other words, one solution dominates the other) so we first work from deterministic initial conditions f and g satisfying $f \geq g$. We also need a growth condition on f and g.

If f does not dominate g, we use the above method to construct a coupling of a pair u, v with initial conditions f and $f \vee g$ which coincide on a finite interval with high probability, and then a separate coupling with initial conditions g and $f \vee g$.

The noises for u and v are chosen as follows. For a fixed parameter $K > 0$, let α and β be

$$\alpha(U) = \sqrt{\frac{(K - |U|)^+}{K}} \qquad \beta(U) = \sqrt{\frac{|U| \wedge K}{K}}. \tag{2.5}$$

By Proposition 1.5.1, we can construct a pair (u, v) which solves the system of equations

$$\left.\begin{aligned} \partial_t u &= \Delta u + a(u) + b(u)\, dW_1 \\ \partial_t v &= \Delta v + a(v) + b(v)[\alpha(U)\, dW_1 + \beta(U)\, dW_2] \end{aligned}\right\} \tag{2.6}$$

for some pair of independent white noises W_1 and W_2, such that $u(0) = f$, $v(0) = g$ and $u \geq v$ with probability one. Note that since $\alpha^2 + \beta^2 = 1$, v is also a solution of (2.1) with respect to the white noise $\alpha(U)W_1 + \beta(U)W_2$. The reasoning behind this particular choice of α and β is that when v is close to u, the noise term of v is almost the same as that of u, which we need if we expect the comparison $u \geq v$ to hold. When the difference U between u and v is large, there is a large degree of independence between the noise terms of u and v, which will enable u and v to come closer together.

The noise term in the equation for U is

$$M = (b(u) - \alpha(U)b(v))W_1 - \beta(U)b(v)W_2$$

so the covariance measure $Q_M(dx\,dy\,ds)$ is

$$\begin{aligned}
&((b(u)-\alpha(U)b(v))^2+\beta(U)^2b(v)^2)\,\delta_{\{x=y\}}\,dx\,dy\,ds\\
&\quad=\left((b(u)-b(v))^2+\frac{2b(u)b(v)(U\wedge K)}{\sqrt{K}(\sqrt{K}+((K-U)^+)^{\frac{1}{2}})}\right)\delta_{\{x=y\}}\,dx\,dy\,ds.
\end{aligned}$$

Hence by following a procedure similar to that in Section 1.5, we can construct another white noise W such that U is a solution of

$$\partial_t U=\Delta U+(a(u)-a(v))+\left[(b(u)-b(v))^2+\frac{2b(u)b(v)(U\wedge K)}{\sqrt{K}(\sqrt{K}+((K-U)^+)^{\frac{1}{2}})}\right]^{\frac{1}{2}}dW.$$

Since the noise term has coefficient at least $(1/L\sqrt{K})(U\wedge K)^{1/2}$ and $a(u)-a(v)$ is negative, this is comparable to the equation $\partial_t U=\Delta U+C\sqrt{U\wedge K}\,dW$ for some constant C dependent on K. It is known that this equation without the noise cut-off at K exhibits local extinction.

Mueller's choice of noises for his coupled system is similar, except in his proof it is sufficient to fix $K=1$ and so have a single pair of solutions which couple with probability one. In our case, we need to use K as a handle to control the error term in our approximation which arises from cutting the noise coefficient off at K, so instead of a single coupling, we need to increase K to increase the probability of coupling.

2.2 Moment bounds on solutions

We now show that for an important class of functions a, condition 4 in Section 2.1 holds.

Lemma 2.2.1 *Suppose f satisfies (2.2), and u is the solution of (1.1) with a and b satisfying conditions 1–3 in Section 2.1. Suppose also that $a(u) = \bar{a}(u) - \theta u$ for $\theta > 0$ and $\bar{a}$ bounded. Then (2.3) holds.*

Proof We may assume without loss of generality that $|\bar{a}|$ is also bounded by L. By the alternate Green's function representation in Lemma 1.1.7, we see that for some $C = C(p)$,

$$
\begin{aligned}
\mathbb{E}[|u(t,x)|^p] &\leq C \int |f(y+x)|^p G^\theta(t,y)\,dy \\
&+ C\mathbb{E}\left[\left(\int_0^t \int |\bar{a}(u(s,y))| G^\theta(t-s,x,y)\,dy\,ds\right)^p\right] \\
&+ C\mathbb{E}\left[\left(\int_0^t \int b(u(s,y))^2 G^\theta(t-s,x,y)^2\,dy\,ds\right)^{p/2}\right] \\
&\leq CK^p \int (1+|x+y|^\gamma)^p G^\theta(t,y)\,dy \\
&+ CL^p \left(\int_0^\infty \int G^\theta(s,x,y)\,dy\,ds\right)^p \\
&+ CL^p \left(\int_0^\infty \int G^\theta(s,x,y)^2\,dy\,ds\right)^{p/2}.
\end{aligned}
$$

It is clear that the last two terms here are bounded. Finally,

$$
\int (1+|x+y|^\gamma)^p G^\theta(t,y)\,dy \leq C(1+|x|^{p\gamma}) \int (1+|y|^{p\gamma}) G^\theta(t,y)\,dy
$$

and the integral here is bounded independently of t. □

We also prove the following estimate on the moments of increments we will need in the coupling proof.

Lemma 2.2.2 *Suppose u is a C_{tem}-valued continuous solution of (2.1), with initial condition f satisfying (2.2), such that (2.3) holds. Then for each $p \geq 2$ there exists a constant C such that for all $x \in \mathbb{R}$,*

$$\sup_{t \geq 1} \sup_{y_1, y_2 \in B_1(x)} \frac{\mathbb{E}\left[|u(t,y_1) - u(t,y_2)|^p\right]}{|y_1 - y_2|^{p/2}} \leq C(1 + |x|^{p\gamma}).$$

Proof Set $\theta = 1$ and let $h = |y_1 - y_2|$. By Lemma 1.1.7, for some constant $C = C(p, \gamma)$,

$$\begin{aligned}
&\mathbb{E}\left[|u(t,y_1) - u(t,y_2)|^p\right] \\
&\leq \; CK^p(1 + |x|^{p\gamma}) \left(\int (1 + |z|^{p\gamma}) \left|G^\theta(t, z+h) - G^\theta(t,z)\right| \, dz \right)^p \\
&+ \; C\mathbb{E}\Bigg[\bigg(\int_0^t \int |a(u(s,z)) + \theta u(s,z)| \times \\
&\qquad\qquad \left|G^\theta(t-s, y_1, z) - G^\theta(t-s, y_2, z)\right| \, dz \, ds \bigg)^p \Bigg] \\
&+ \; CL^p \left(\int_0^t \int \left(G^\theta(t-s, y_1, z) - G^\theta(t-s, y_2, z)\right)^2 \, dz \, ds \right)^{p/2}.
\end{aligned}$$

We control the first term by the Mean Value Theorem as follows.

$$\left|G^\theta(t, z+h) - G^\theta(t,z)\right| \leq \frac{he^{-\theta t}}{2t\sqrt{4\pi t}} \sup_{w \in (z, z+h)} |w| e^{-w^2/4t}.$$

Since

$$\sup_{w \in (z, z+h)} e^{-w^2/4t} \leq \begin{cases} e^{-z/16t} & \text{if } |z| \geq 2 \\ 1 \leq e^{1/4} e^{-z^2/16t} & \text{if } |z| \leq 2 \text{ and } t \geq 1 \end{cases}$$

we have, for $z \in \mathbb{R}$ and $t \geq 1$,

$$\left|G^\theta(t, z+h) - G^\theta(t,z)\right| \leq \frac{h(|z|+1)e^{1/4}e^{-\theta t}e^{-z^2/16t}}{2t\sqrt{4\pi t}}.$$

It is now straightforward to show that

$$\sup_{t\geq 1}\int (1+|z|^{p\gamma})\left|G^\theta(t,z+h)-G^\theta(s,z)\right|\,dz \leq Ch$$

for some $C=C(\theta,p,\gamma)$. The second term is bounded by a constant times

$$\sup_{s\geq 0,z\in\mathbb{R}}\mathbb{E}\left[\frac{|u(s,z)|^p+1}{1+|z|^{p\gamma}}\right]\left(\int_0^\infty\int (1+|z|^\gamma)\left|G^\theta(s,y_1,z)-G^\theta(s,y_2,z)\right|\,dz\,ds\right)^p$$

and the expectation here is bounded by hypothesis. To deal with the integral, we note that

$$\begin{aligned}
&\left|G^\theta(s,y_1,z)-G^\theta(s,y_2,z)\right| \\
&= \sqrt{2}\left(e^{-(y_1-z)^2/8s}+e^{-(y_2-z)^2/8s}\right)\left|G^\theta(2s,y_1,z)-G^\theta(2s,y_2,z)\right|
\end{aligned}$$

and for $y\in B_1(x)$,

$$\begin{aligned}
(1+|z|^\gamma)\,e^{-(y-z)^2/8s} &\leq 3^\gamma\left(1+|z-y|^\gamma+|y-x|^\gamma+|x|^\gamma\right)e^{-(y-z)^2/8s} \\
&\leq C(1+s^{\gamma/2})(1+|x|^\gamma)
\end{aligned}$$

for some $C=C(\gamma)$ (where we control $|z-y|^\gamma e^{-(z-y)^2/8s}$ by minimising over all $|z-y|\geq 0$). We then apply Lemma 1.3.3 to control the integral. The third term above is easy to control, hence the result follows. □

By applying the Garsia-Rodemich lemma (see Lemma 1.3.2), we see that for $p\geq 8$ there exists a constant C such that for all $x\in\mathbb{R}$,

$$\sup_{t\geq 1}\mathbb{E}\left[\sup_{y_1,y_2\in B_1(x)}\frac{|u(t,y_1)-u(t,y_2)|^p}{|y_1-y_2|^{p/4}}\right]\leq C(1+|x|^{p\gamma}). \tag{2.7}$$

2.3 PDE results

In this section, we prove results about a PDE we need for the main coupling result. Fix $\phi \in C_c^+(\mathbb{R})$ and let ξ be the unique nonnegative solution of

$$\begin{aligned} \partial_t \xi &= \Delta \xi - \xi^2 \\ \xi(0) &= \phi. \end{aligned} \tag{2.8}$$

We have the following Green's function representation of ξ:

$$\xi(t,x) = \int \phi(y) G(t, x-y)\, dy - \int_0^t \int \xi(s,y)^2 G(t-s, x-y)\, dy\, ds. \tag{2.9}$$

Lemma 2.3.1 *For some $C = C(\phi, t)$,*

$$\sup_{s \le t} \xi(s,x) \le C e^{-x^2/16t}.$$

Proof Choose $R > 0$ such that supp $\phi \subseteq [-R, R]$. For $|x| \ge 4R$ and $0 < s \le t$,

$$\begin{aligned} \xi(s,x) &\le \int \phi(y) G(s, x-y)\, dy \\ &\le \left(\sup_{y \in \mathbb{R}} \phi(y) \right) \int_{-R}^{R} G(s, x-y)\, dy \\ &\le \left(2R \sup_{y \in \mathbb{R}} \phi(y) \right) \frac{e^{-(|x|-R)^2/4s}}{\sqrt{4\pi s}} \\ &= \left(2R \sup_{y \in \mathbb{R}} \phi(y) \right) \frac{e^{-((|x|-2R)+R)^2/4s}}{\sqrt{4\pi s}} \\ &\le 2R \sup_{y \in \mathbb{R}} \phi(y) \left(\frac{e^{-R^2/4s}}{\sqrt{4\pi s}} \right) e^{-x^2/16t}. \end{aligned}$$

Note that $e^{-R^2/4s}/\sqrt{4\pi s}$ is bounded in $s > 0$. Now for $|x| < 4R$ or $s = 0$,

$$\xi(s,x) \le \sup_{y \in \mathbb{R}} \phi(y) \le (\sup_{y \in \mathbb{R}} \phi(y)) e^{R^2/t} e^{-x^2/16t}$$

which gives the result. $\square$

We will also need an estimate on $|\partial_x \xi(t,x)|$. The following is based on Lemma 4 in Tribe [30], but with more work to get the explicit dependence on x.

Lemma 2.3.2 *Suppose supp* $\phi \subseteq [-R, R]$ *for* $R > 0$. *Then for* $|x| \geq 4R$ *we have, for some* $C = C(\phi, t)$,

$$\sup_{s \leq t} |\partial_x \xi(s,x)| \leq C e^{-x^2/20t}.$$

Proof Checking the conditions necessary to pass the derivative through the integrals, we see that

$$\begin{aligned}\sup_{s\leq t} |\partial_x \xi(s,x)| &\leq \sup_{s \leq t} \int \phi(y)\, |\partial_x G(s,x,y)|\, dy + \\ &\quad \int_0^t \int \xi(t-s,y)^2\, |\partial_x G(s,x,y)|\, dy\, ds.\end{aligned}$$

Label the terms on the right I_1 and I_2 respectively. Now, since $|z| \leq e^{z^2+1}$ for all real z, we have, for all $s \leq t$,

$$|\partial_x G(s,x,y)| = \frac{|x-y| e^{-(x-y)^2/4s}}{4s\sqrt{\pi s}} \leq \frac{Ce^{-(x-y)^2/5s}}{s}$$

where throughout the proof, C stands for a constant depending only on ϕ and t whose value may change from line to line.

$$\begin{aligned} I_1 &\leq C \sup_{s\leq t} \int_{-R}^{R} \frac{e^{-(x-y)^2/5s}}{s}\, dy \\ &\leq C \sup_{s \leq t} \frac{e^{-(|x|-2R)^2/5s} e^{-R^2/5s}}{s} \\ &\leq C \left(\sup_{r>0} \frac{e^{-R^2/5r}}{r} \right) e^{-x^2/20t}. \end{aligned}$$

Applying Lemma 2.3.1,

$$\begin{aligned}
I_2 &\le C\int_0^t\int \frac{e^{-(x-y)^2/8t}e^{-y^2/5s}}{s}\,dy\,ds \\
&\le C\int_0^t\int \frac{e^{-(x-y)^2/8t}e^{-y^2/10t}e^{-y^2/10s}}{s}\,dy\,ds \\
&\le C\left(\int e^{-(x-y)^2/4t}e^{-y^2/5t}\,dy\right)^{\frac{1}{2}}\int_0^t \frac{1}{s}\left(\int e^{-y^2/5s}\,dy\right)^{\frac{1}{2}}\,ds
\end{aligned}$$

by the Cauchy-Schwartz inequality. Now the integral in s is just a constant times $\int_0^t s^{-3/4}\,ds$ which is finite. Finally,

$$\begin{aligned}
&\int e^{-(x-y)^2/4t}e^{-y^2/5t}\,dy \\
&= \int_{x-\frac{|x|}{2}}^{x+\frac{|x|}{2}} e^{-(x-y)^2/4t}e^{-y^2/5t}\,dy + \int_{|x-y|\ge\frac{|x|}{2}} e^{-(x-y)^2/4t}e^{-y^2/5t}\,dy \\
&\le 2\left(\int e^{-y^2/5t}\right)e^{-x^2/20t}
\end{aligned}$$

as required. □

Here, we give a modified version of the parabolic Weak Maximum Principle given in Renardy and Rogers [26].

Lemma 2.3.3 *Suppose* $\psi : [0,T]\times[-N,N] \to \mathbb{R}$ *is continuous in* t *and* x, *and* $\Delta\psi(t,x)$ *and* $\partial_t\psi(t,x)$ *exist and are continuous on* $\Omega := (0,T]\times(-N,N)$. *Suppose also that* $c : \overline{\Omega} \to \mathbb{R}$ *is continuous and nonpositive, and*

$$-\partial_t\psi + \Delta\psi + c(t,x)\psi \le 0$$

on Ω. *Then*

$$\inf\psi \ge \left(\inf_{(t,x)\in\overline{\Omega}\setminus\Omega}\psi(t,x)\right)\wedge 0.$$

Proof Suppose $-\partial_t\psi + \Delta\psi + c(t,x)\psi < 0$ on Ω and ψ has a minimum at $(t,x) \in \Omega$. Then $\Delta\psi(t,x) \geq 0$ and $\partial_t\psi(t,x) \leq 0$, so $c(t,x)\psi(t,x)$ must be negative and hence $\psi(t,x)$ must be positive. Now choose γ such that $\gamma^2 > \sup|c|$ and set $\psi_\varepsilon = \psi - \varepsilon e^{\gamma x}$ for $\varepsilon > 0$. Then

$$-\partial_t\psi_\varepsilon + \Delta\psi_\varepsilon + c(t,x)\psi_\varepsilon = -\partial_t\psi + \Delta\psi + c(t,x)\psi(t,x) - \varepsilon e^{\gamma x}\left(\gamma^2 + c(t,x)\right)$$

which is strictly negative, so by above,

$$\inf\psi \geq \inf\psi_\varepsilon \geq \left(\inf_{(t,x)\in\overline{\Omega}\setminus\Omega}\psi_\varepsilon(t,x)\right) \wedge 0.$$

Taking $\varepsilon \to 0$ gives the result. □

Lemma 2.3.4 *Let $f, g : \mathbb{R} \to \mathbb{R}^+$ be bounded continuous functions with $f \geq g$, and let ξ^f, ξ^g be solutions of (2.8) with $\xi^f(0) = f$ and $\xi^g(0) = g$. Suppose that g is compactly supported. Then $\xi^f \geq \xi^g$.*

Proof Choose $T > 0$ and $N > 0$. Set $\psi = \xi^f - \xi^g$. Then ψ satisfies

$$\partial_t\psi(t,x) = \Delta\psi(t,x) + c(t,x)\psi$$

for all $t > 0$, $x \in \mathbb{R}$, where $c(t,x) = -(\xi^f + \xi^g)(t,x) \leq 0$. By Lemma 2.3.3,

$$\begin{aligned}\inf_{(s,x)\in[0,T]\times[-N,N]}\psi(s,x) &\geq \left(\inf_{0<s\leq T}\psi(s,\pm N)\right) \wedge 0 \\ &\geq -\sup_{0<s\leq T}\xi^g(s,\pm N).\end{aligned}$$

By Lemma 2.3.1, this is bounded below by $-Ce^{-N^2/16T}$ for some $C = C(g,T)$. Taking $N \to \infty$, we see that $\psi(s,x) \geq 0$ for $0 \leq s \leq T$ and $x \in \mathbb{R}$. Since we can take T arbitrarily large, the result follows. □

Lemma 2.3.5 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be compactly supported and nonnegative, and let ξ^f, ξ^g and ξ^{f+g} be solutions of (2.8) with initial conditions f, g and $f+g$ respectively. Then*

$$\xi^f + \xi^g \geq \xi^{f+g}.$$

Proof Set $\psi = \xi^f + \xi^g - \xi^{f+g}$ and $c = -(\xi^f + \xi^g + \xi^{f+g})$. Then

$$-\partial_t \psi + \Delta \psi + c(t,x)\psi = -2\xi^f \xi^g \leq 0$$

so applying Lemma 2.3.3, we see that for any $N > 0$ and $T > 0$,

$$\begin{aligned} \inf_{(s,x)\in[0,T]\times[-N,N]} \psi(s,x) &\geq \left(\inf_{0<s\leq T} \psi(s, \pm N) \right) \wedge 0 \\ &\geq -\sup_{0<s\leq T} \xi^{f+g}(s, \pm N). \end{aligned}$$

Applying Lemma 2.3.1 as above gives the result. □

Lemma 2.3.6 *Suppose $f \in C_c^1(\mathbb{R})$ is nonnegative, symmetric about 0 and nondecreasing on $(-\infty, 0]$. Let ξ^f be the non-negative solution of (2.8) with initial condition f. Then for each $t > 0$, $\xi(t)$ is also symmetric about 0 and nondecreasing for $x \in (-\infty, 0]$ (and hence is unimodal).*

Proof Symmetry follows trivially from uniqueness since $\xi^f(t, -x)$ is also a solution of (2.8) with $\xi^f(0,x) = \xi^f(0,-x)$, so $\xi^f(t,x) = \xi^f(t,-x)$.

To prove unimodality, set $\psi(t,x) = \partial_x \xi(t,x)$. Then ψ satisfies

$$\begin{aligned} \partial_t \psi &= \Delta \psi - 2\xi\psi \\ \psi(t,0) &= 0 \text{ for } t \geq 0 \\ \psi(0,x) &\geq 0 \text{ for } x \leq 0. \end{aligned}$$

Applying Lemma 2.3.3, we see that for $T > 0$ and $N > 0$,

$$\inf_{(t,x)\in[0,T]\times[-N,0]} \psi(t,x) \geq \left(\inf_{t\in[0,T]} \psi(t,-N)\right) \wedge 0 \geq - \sup_{t\in[0,T]} |\psi(t,-N)|$$

which converges to zero for each T as $N \to \infty$ by Lemma 2.3.2. Hence $\psi(t,x)$ is nonnegative for each $x \leq 0$ and so for each t, $\xi(t,x)$ is nondecreasing for $x \leq 0$. $\square$

The estimate on solutions given by Lemma 2.3.1 is obtained by throwing away the $-\xi^2$ term and considering a linear equation. However, this estimate blows up if we take the initial condition to be large since we don't have the $-\xi^2$ term to pull solutions down in finite time. The following estimate is obtained by rescaling Lemma 3.1 of Dawson, Iscoe and Perkins [4] and doesn't depend on the height of the initial condition.

Lemma 2.3.7 *Fix $\phi \in C_c^+(\mathbb{R})$ such that supp $\phi \subseteq [-R, R]$, and let ξ be the non-negative solution of (2.8). Then for all $t > 0$, $x \in \mathbb{R}$ we have $\xi(t,x) \leq 1/t$. Furthermore, for $|x| \geq 2(R \vee t^{1/2})$ we have*

$$\xi(t,x) \leq \frac{24e}{t} e^{-|x|/2t^{1/2}}.$$

Proof To get $\xi(t,x) \leq 1/t$, compare the solution to one with constant initial condition, using Lemma 2.3.4.

Let $\psi(t,x) = R^2\xi(R^2t/2, Rx)$. Then $\partial_t\psi = \frac{1}{2}(\Delta\psi - \psi^2)$ and supp $\psi(0) \subseteq [-1, 1]$. Applying the result in [4] gives, for $t > 0$ and $|x| \geq 2$,

$$\psi(t,x) \leq \left(\frac{12e}{t}\right) \frac{e^{-|x|/\sqrt{2t}}}{(1 - e^{-|x|/\sqrt{2t}})^2}.$$

When $|x| \geq t^{1/2}$ we have $1 - e^{-|x|/\sqrt{2t}} \geq 1/2$, so when $|x| \geq 2 \vee t^{1/2}$,

$$\psi(t, x) \leq \frac{48e}{t} e^{-|x|/\sqrt{2t}}.$$

Since $\xi(t, x) = R^{-2}\psi(2t/R^2, x/R)$, when $|x/R| \geq 2 \vee \sqrt{2t/R^2}$,

$$\xi(t, x) \leq \frac{24e}{t} e^{-|x|/2t^{1/2}}.$$

The condition on $|x|$ holds when $|x| \geq 2(R \vee t^{1/2})$, as required. □

The following is a slight simplification for small t.

Corollary 2.3.8 *Suppose ξ and R are as in Lemma 2.3.7. Suppose $0 < t \leq 1$ and $|x| \geq 3 \vee 2R$. Then*

$$\xi(t, x) \leq (24e)e^{-|x|/4t^{1/2}} \leq (24e)e^{-|x|/4}$$

Proof By the previous lemma,

$$\xi(t, x) \leq (24e) \left(\frac{e^{-|x|/4t^{1/2}}}{t} \right) e^{-|x|/4t^{1/2}}.$$

Now $e^{-|x|/4t^{1/2}}/t \leq 1$ when $|x| \geq 4t^{1/2} \log(t^{-1})$. It is easy to see that $t^{1/2} \log(t^{-1})$ is bounded above by $2/e < 3/4$ for $t \in (0, 1]$, hence the result. □

Lemmas 2.3.5 and 2.3.6 allow us to prove a result approximating the area under a solution of (2.8), which will be crucial later on. We need to choose a special initial condition ϕ_0 for the following proof to work. Let $\phi_0 \in C_c^1(\mathbb{R})^+$ be unimodal, symmetric about 0 and such that

$$\phi_0(x) = \begin{cases} 0 & \text{if } |x| \geq R \\ > 0 & \text{if } |x| < R \\ 1 & \text{if } |x| \leq R/2. \end{cases} \tag{2.10}$$

Lemma 2.3.9 *Fix any $\alpha > 0$. Suppose ξ is the nonnegative solution of (2.8) with $\xi(0) = \alpha\phi_0$. Then for any $t \geq 0$, $x \in \mathbb{R}$ and $L \in [0, R/2]$ we have*

$$\int_{x-L}^{x+L} \xi(t, y)\, dy \geq \frac{L}{4}\xi(t, x).$$

Proof We first deal with the case $x = 0$. Let ϕ_1 and ϕ_2 be the following functions.

$$\phi_1(y) = \begin{cases} \phi_0(y) & \text{if } y \leq 0 \\ \phi_0(y + \frac{R}{2}) & \text{if } y > 0 \end{cases}$$

$$\phi_2(y) = \begin{cases} \phi_0(y - \frac{R}{2}) & \text{if } y \leq 0 \\ \phi_0(y) & \text{if } y > 0. \end{cases}$$

Clearly, $\phi_1, \phi_2 \leq \phi_0 \leq \phi_1 + \phi_2$ and ϕ_1, ϕ_2 are symmetric around $-R/4$ and $R/4$ respectively. Let ξ_i be the nonnegative solution of (2.8) with $\xi_i(0) = \alpha\phi_i$ for $i = 1, 2$. Suppose $L \in [0, R/4]$. For $y \in [-L, 0]$, $\xi_1(t, y) \geq \xi_1(t, 0)$ by unimodality. Similarly, for $y \in [0, L]$, $\xi_2(t, y) \geq \xi_2(t, 0)$. Hence

$$\int_{-L}^{L} (\xi_1 + \xi_2)(t, y)\, dy \geq L(\xi_1 + \xi_2)(t, 0).$$

By monotonicity and sublinearity, we have $2\xi \geq \xi_1 + \xi_2 \geq \xi$, so

$$\int_{-L}^{L} \xi(t, y)\, dy \geq \frac{L}{2}\xi(t, 0).$$

We now consider the general case. Suppose $x \in \mathbb{R}$ and $L \leq R/2$. If $|x| \geq L/2$ the result follows from unimodality, so we may assume that $|x| \leq L/2$. Then $[x - L, x + L] \supseteq [-L/2, L/2]$ so applying the special case above, we see that

$$\int_{x-L}^{x+L} \xi(t, y)\, dy \geq \int_{-L/2}^{L/2} \xi(t, y)\, dy \geq \frac{L}{4}\xi(t, 0) \geq \frac{L}{4}\xi(t, x)$$

as required. □

2.4 Main coupling result

Theorem 2.4.1 *Fix any $R > 0$ and $\varepsilon > 0$. Suppose $f, g \in C(\mathbb{R})$ satisfy*

$$\sup_{y\in\mathbb{R}} \left(\frac{|f(y)|}{1+|y|^\gamma} \right) + \sup_{y\in\mathbb{R}} \left(\frac{|g(y)|}{1+|y|^\gamma} \right) < \infty \tag{2.11}$$

for some $\gamma \in (0, 1/3)$, and $f \geq g$. Then there exists a probability space containing a pair of C_{tem}-valued continuous solutions (u, v) of (2.1) with $(u(0), v(0)) = (f, g)$ such that for sufficiently large $t > 0$,

$$\mathbb{P}\left((u-v)(t)\chi_{(-R,R)} = 0\right) \geq 1 - \varepsilon.$$

Let (u^K, v^K) be a solution of the system (2.6) described in the introduction, where α and β are defined in (2.5). We show that given ε and R, this pair satisfies the requirement of the above theorem for suitably large K.

To simplify notation, we drop the superscript. Note that the distributions of u^K and v^K are each independent of K. Recall that we define $U = u - v$, which is almost surely nonnegative. The distribution of U will depend on K in general, but we can bound moments by noting that $U^n \leq 2^n(|u|^n + |v|^n)$ for $n \in \mathbb{N}$.

The first step in the proof is to derive an approximation for the death probability in terms of an exponential expectation and an error term which arises from cutting off U at K in the noise term.

Let $\phi_0 \in C_c^\infty(\mathbb{R})$ be unimodal, symmetric around 0 and such that (2.10) holds. For each $\mu > 0$, let ξ^μ be the unique nonnegative solution of (2.8) with $\xi^\mu(0) = \mu\phi_0$.

Lemma 2.4.2 *Suppose $K \geq 1$ and $t \geq 2$. Then*

$$\mathbb{P}\left(U(t)\chi_{(-R,R)} = 0\right) \geq \lim_{\mu\to\infty} \mathbb{E}\left[e^{-2KL^2(f-g,\xi^\mu(t))}\right] - L^2\left(Er_1(K) + Er_2(K)\right),$$

where

$$
\begin{aligned}
Er_1(K) &= \limsup_{\mu\to\infty} \frac{1}{K}\int_1^\infty \int \xi^\mu(s,x)^2 \sup_{r\geq 0}\mathbb{E}\left[U(r,x)^3\right]\,dx\,ds\\
Er_2(K) &= \\
&\limsup_{\mu\to\infty} \frac{1}{K}\int_0^1 \int \sup_{r\geq 1}\mathbb{E}\left[\xi^\mu(s,x)^2 U(r,x)^3 \chi_{[U(r,x)\geq K]} e^{-K(U(r),\xi^\mu(s))}\right]\,dx\,ds.
\end{aligned}
$$

Proof First, note that

$$
\mathbb{P}\left(U(t)\chi_{(-R,R)}=0\right) = \lim_{\mu\to\infty}\mathbb{E}\left[e^{-\mu(U(t),\phi_0)}\right].
$$

Suppose that $\psi : [0,t]\times\mathbb{R}\to\mathbb{R}^+$ is a test function, as described in Chapter 1. Then, for some martingale M,

$$
\begin{aligned}
d(U(t),\psi(t)) &= (U(t),(\Delta+\partial_t)\psi(t))\,dt + (a(u(t))-a(v(t)),\psi(t))\,dt + dM_t\\
&\leq (U(t),(\Delta+\partial_t)\psi(t))\,dt + dM_t,\\
d\left\langle (U,\psi)\right\rangle_t &= \int \psi(t,y)^2\left[(b(u)-b(v))^2 + \frac{2b(u)b(v)(U\wedge K)}{K(1+\alpha(U))}\right]dy\,dt.
\end{aligned}
$$

Since b is bounded below by $1/L$ and $U\wedge K = U-(U-K)^+$, we have

$$
d\left\langle (U,\psi)\right\rangle_t \geq \frac{1}{KL^2}\left(\int \psi(t,y)^2 U(t,y)\,dy\,dt - \int \psi(t,y)^2\,(U-K)^+\,dy\,dt\right).
$$

Applying Itô's formula, we see that

$$
\begin{aligned}
e^{-(U(t),\psi(t))} - e^{-(f-g,\psi(0))} &= \int_0^t e^{-(U(s),\psi(s))}\left(\frac{1}{2}d\left\langle (U,\psi)\right\rangle_s - d(U(s),\psi(s))\right)\\
&\geq \int_0^t e^{-(U(s),\psi(s))}\left(U(s),\left(\frac{1}{2KL^2}\psi^2-\Delta\psi-\partial_t\psi\right)(s)\right)ds - \\
&\quad \frac{1}{2KL^2}\int_0^t e^{-(U(s),\psi(s))}\int \psi(s,y)^2\,(U(s,y)-K)^+\,dy\,ds + \\
&\quad \int_0^t e^{-(U(s),\psi(s))}\,dM_s.
\end{aligned}
$$

Since $\psi \geq 0$, it is clear that the stochastic integral term is a martingale. We now choose

$$\psi(s) = 2KL^2\xi^{\mu/2KL^2}(t-s).$$

Then $\psi(t) = \mu\phi_0$ and

$$(1/2KL^2)\psi^2 - \Delta\psi - \partial_t\psi = 0.$$

To see that ψ is a test function, note that since $\Delta\psi + \partial_t\psi$ is a constant times ψ^2 and ψ is bounded, we can replace $\Delta\psi + \partial_t\psi$ by ψ in (1.13). (1.12) and (1.13) then follow from Lemma 2.3.2, and (1.14) follows from (2.9).

For convenience, we set $\xi^{\mu,K} = \xi^{\mu/2KL^2}$. So

$$\mathbb{E}\left[e^{-\mu(U(t),\phi_0)}\right] \geq \mathbb{E}\left[e^{-2KL^2(\xi^{\mu,K}(t),f-g)}\right] -$$
$$2KL^2\int_0^t\int \mathbb{E}\left[\xi^{\mu,K}(s)^2 e^{-2KL^2(U(t-s),\xi^{\mu,K}(s))}\left(U(t-s)-K\right)^+\right]\,dy\,ds.$$

Now note that for any $U \geq 0$, $K > 0$ we have, by a short calculation,

$$(U-K)^+ \leq (4U^3/27K^2)\chi_{(U\geq K)}.$$

Also, $L \geq 1$ so we can throw away the $2L^2$ on the exponent of the error term. We plug these into the above expression. Taking $\mu \to \infty$, $\mu/2KL^2$ also goes to infinity, so we can replace $\xi^{\mu,K}$ with ξ^{μ}. This gives the required result. □

We show that for each K, for t large enough,

$$\lim_{\mu\to\infty}\mathbb{E}\left[e^{-2KL^2(f-g,\xi^{\mu}(t))}\right]$$

is arbitrarily close to one, and that we can choose K large enough to make $\mathrm{Er}_1(K)$ and $\mathrm{Er}_2(K)$ arbitrarily small. Theorem 2.4.1 then follows.

Lemma 2.4.3 *For each $K > 0$,*

$$\lim_{t\to\infty}\lim_{\mu\to\infty}\mathbb{E}\left[e^{-2KL^2(f-g,\xi^\mu(t))}\right] = 1.$$

Proof For each $t > 0$, $x \in \mathbb{R}$, we set

$$\xi(t,x) = \lim_{\mu\to\infty}\xi^\mu(t,x).$$

This limit exists since ξ^μ is increasing in μ, and by Lemma 2.3.7, the limit is finite for $t > 0$. Also,

$$\lim_{\mu\to\infty}\mathbb{E}\left[e^{-2KL^2(f-g,\xi^\mu(t))}\right] = \mathbb{E}\left[e^{-2KL^2(f-g,\xi(t))}\right].$$

By dominated convergence and the hypotheses on f and g, it suffices to show that for $\gamma \in (0, 1/3)$,

$$\lim_{t\to\infty}\int(1+|x|^\gamma)\xi(t,x)\,dx = 0.$$

By Lemma 2.3.7, for $t \geq R^2 \vee 1$ we have

$$\begin{aligned}
&\int(1+|x|^\gamma)\xi(t,x)\,dx \\
&= \int_{-2\sqrt{t}}^{2\sqrt{t}}(1+|x|^\gamma)\xi(t,x)\,dx + \int_{|x|\geq 2\sqrt{t}}(1+|x|^\gamma)\xi(t,x)\,dx \\
&\leq \frac{4}{\sqrt{t}}(1+(2t^{1/2})^\gamma) + \frac{48e}{t}\int|x|^\gamma e^{-|x|/\sqrt{4t}}\,dx \\
&\leq 2^{\gamma+2}t^{(\gamma-1)/2} + 4t^{-1/2} + (48e)t^{(\gamma-1)/2}\int|x|^\gamma e^{-|x|/2}\,dx.
\end{aligned}$$

This clearly converges to zero whenever $0 < \gamma < 1$. □

This deals with the main term in our approximation. To complete the result, we need to control the error terms.

Lemma 2.4.4

$$\lim_{K\to\infty} Er_1(K) = 0.$$

Proof By (2.3) we know that for each $x \in \mathbb{R}$,

$$\sup_{r\geq 0} \mathbb{E}\left[U(r,x)^3\right] \leq 8 \sup_{r\geq 0} \mathbb{E}\left[|u(r,x)|^3 + |v(r,x)|^3\right] \leq C(1+|x|^{3\gamma})$$

for some $C = C(f,g,a,b)$ and $\gamma \in (0, 1/3)$. By Lemma 2.3.7,

$$\begin{aligned}\int \sup_{r\geq 0} \mathbb{E}\left[U(r,x)^3\right] \xi(s,x)^2\, dx &\leq \frac{4C\,(R\vee\sqrt{s})\,(1+(2(R\vee\sqrt{s}))^{3\gamma})}{s^2} \\ &+ \frac{(24e)^2 C}{s^2} \int e^{-|x|/\sqrt{s}} \left(1+|x|^{3\gamma}\right) dx \\ &= \frac{4C\,(R\vee\sqrt{s})\,(1+(2(R\vee\sqrt{s}))^{3\gamma})}{s^2} \\ &+ \frac{(24e)^2 C}{s^{3/2}} \int e^{-|z|} \left(1+|z\sqrt{s}|^{3\gamma}\right) dz\end{aligned}$$

which is integrable over $s \in [1,\infty)$. Hence $\mathrm{Er}_1(K)$ is bounded by a constant times $1/K$. □

Lemma 2.4.5

$$\lim_{K\to\infty} Er_2(K) = 0.$$

Proof We prove this by splitting the integral into two regions and estimating each separately. For $A \in \mathcal{B}(\mathbb{R})$, set

$$\mathrm{Er}_2(K,A) \;=\; \limsup_{\mu\to\infty} \frac{1}{K} \int_0^1 \int_A \sup_{r\geq 1} \mathbb{E}\left[\xi^\mu(s,x)^2 U(r,x)^3 \chi_{[U(r,x)\geq K]} e^{-K(U(r),\xi^\mu(s))}\right] dx\, ds.$$

Set $A_2 = [-(3 \vee 2R), 3 \vee 2R]$ and $A_1 = \mathbb{R} \setminus A_2$. Then by Corollary 2.3.8,

$$\begin{aligned} \mathrm{Er}_2(K, A_1) &\leq \frac{(24e)^2}{K} \int_0^1 \int e^{-|x|/2} \sup_{r \geq 1} \mathbb{E}\left[U(r,x)^3\right] \, dx \, ds \\ &\leq \frac{C}{K} \int e^{-|x|/2} \left(1 + |x|^{3\gamma}\right) \, dx \end{aligned}$$

for some constant C depending only on f, g, a and b, and $\gamma \in (0, 1/3)$. Hence $\lim_{K \to \infty} \mathrm{Er}_2(K, A_1) = 0$.

It now remains to show convergence on a space-time block close to the origin. This is a little trickier, since it is here that $\xi(s,x)$ blows up. The idea is to use modulus of continuity estimates on U to show that if $U(r,x) \geq K$ then $U(r,y)$ will be at least $K - 1$ on a patch around x which is not too small. This allows us to use the exponential term to control the integrand.

For each $x \in \mathbb{R}$ and $r \geq 1$, we set

$$\Gamma(r,x) = \sup_{y_1, y_2 \in B_1(x), y_1 \neq y_2} \frac{|U(r,y_1) - U(r,y_2)|}{|y_1 - y_2|^{1/4}}$$

and

$$\Pi(r,x) = \min\left\{\frac{1}{\Gamma(r,x)^4}, 1, \frac{R}{2}\right\}.$$

If $|x - y| \leq \Pi(r,x)$ then

$$|x - y| \leq \inf_{y_1, y_2 \in B_1(x)} \frac{|y_1 - y_2|}{|U(r,y_1) - U(r,y_2)|^4} \leq \frac{|x-y|}{|U(r,x) - U(r,y)|^4}$$

and hence

$$|U(r,x) - U(r,y)| \leq 1$$

so on the interval $(x - \Pi(r,x), x + \Pi(r,x))$, $U(r,y)$ doesn't deviate from $U(r,x)$ by more than one. Suppose $K \geq 2$. Then if $U(r,x) \geq K$ we have

$$\int U(r,y)\xi^\mu(s,y)\,dy \geq (K-1) \int_{x-\Pi(r,x)}^{x+\Pi(r,x)} \xi^\mu(s,y)\,dy \geq \frac{K}{2} \int_{x-\Pi(r,x)}^{x+\Pi(r,x)} \xi^\mu(s,y)\,dy.$$

By Lemma 2.3.9, this is bounded below by $K\Pi(r,x)\xi^\mu(s,x)/8$. Hence

$$\xi^\mu(s,x)^2\chi_{[U(r,x)\geq K]}e^{-K(U(r),\xi^\mu(s))} \leq \xi^\mu(s,x)^2\exp\left(\frac{-K^2\Pi(r,x)\xi^\mu(s,x)}{8}\right).$$

A little calculus shows that $X^2e^{-\alpha X} \leq (4/e^2)\alpha^{-2}$ all $X \geq 0$ and $\alpha > 0$, so

$$\xi^\mu(s,x)^2\chi_{[U(r,x)\geq K]}e^{-K(U(r),\xi^\mu(s))} \leq \frac{256}{e^2K^4}\Pi(r,x)^{-2}$$

and hence

$$\begin{aligned}
\mathrm{Er}_2(K,A_2) &\leq \frac{256|A_2|}{e^2K^5}\sup_{r\geq 1, x\in A_2}\mathbb{E}\left[\Pi(r,x)^{-2}U(r,x)^3\right] \\
&\leq \frac{256(8\vee 4R)}{e^2K^5}\sup_{r\geq 1, x\in A_2}\mathbb{E}\left[\left(\Gamma(r,x)^8+1+\frac{4}{R^2}\right)U(r,x)^3\right]
\end{aligned}$$

which we can control by Cauchy-Schwartz, (2.3) and (2.7) since A_2 is bounded. Hence $\mathrm{Er}_2(K,A_2)$ converges to zero as $K\to\infty$ and the proof is complete. □

2.5 Application to uniqueness of the stationary distribution

Now we've proved the coupling result, we can show uniqueness of stationary distributions. For each $n \in \mathbb{N}$, we set

$$\mathcal{F}_n = \{|_{[-n,n]}^{-1}(B) \ : \ B \in \mathcal{B}(C([-n,n]))\}$$

where $|_{[-n,n]} : C_{tem} \to C([-n,n])$ is the restriction map $f \mapsto f|_{[-n,n]}$. Each $\mathcal{F}_n$ is a σ-algebra and since $|_{[-n,n]}$ is continuous, $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{B}(C_{tem})$. We also note that $\mathcal{A} = \bigcup_{n\in\mathbb{N}}\mathcal{F}_n$ is an algebra.

Lemma 2.5.1 *Suppose f and g satisfy (2.11) and for each $t \geq 0$, let μ_t and ν_t be the probability distributions of $u(t)$ and $v(t)$, where u and v are solutions of (2.1) started from f and g respectively. Then, for each $n \in \mathbb{N}$,*

$$\lim_{t\to\infty} \sup_{A\in\mathcal{F}_n} |\mu_t(A) - \nu_t(A)| = 0.$$

Proof We first suppose that $f \geq g$. Suppose $A \in \mathcal{F}_n$. Then for solutions u and v on the same space started from f and g respectively, by splitting the probabilities as in (2.4),

$$|\mu_t(A) - \nu_t(A)| \leq \mathbb{P}\left(u(t)|_{[-n,n]} \neq v(t)|_{[-n,n]}\right)$$

which we can make arbitrarily small by choosing a suitable coupling and t sufficiently large. If f does not dominate g, for each $t \geq 0$ let η_t be the distribution of $w(t)$, where w is a solution of (2.1) with $w(0) = f \vee g$. Then

$$|\mu_t(A) - \nu_t(A)| \leq |\mu_t(A) - \eta_t(A)| + |\eta_t(A) - \nu_t(A)|$$

which converges to zero as before. □

Corollary 2.5.2 *Suppose μ and ν are probability distributions on C_{tem} such that (2.11) is satisfied μ- and ν-almost surely. Then for all $A \in \mathcal{F}_n$,*

$$\lim_{t\to\infty} \sup_{A\in\mathcal{F}_n} \left| \int \mathbb{P}_f(u(t) \in A)\, \mu(df) - \int \mathbb{P}_f(u(t) \in A)\, \nu(df) \right| = 0$$

where $\mathbb{P}_f$ is the law of a solution of (2.1) started from f.

Proof

$$\left| \int \mathbb{P}_f(u(t) \in A)\, \mu(df) - \int \mathbb{P}_f(u(t) \in A)\, \nu(df) \right|$$

$$
\begin{aligned}
&= \left| \int\int \mathbb{P}_f(u(t) \in A)\, \mu(df)\nu(dg) - \int\int \mathbb{P}_g(u(t) \in A)\, \mu(df)\nu(dg) \right| \\
&\leq \int\int |\mathbb{P}_f(u(t) \in A) - \mathbb{P}_g(u(t) \in A)|\, \mu(df)\nu(dg)
\end{aligned}
$$

which converges to zero as $t \to \infty$ by Lemma 2.5.1. □

Theorem 2.5.3 *There is a unique stationary distribution μ on C_{tem} such that (2.11) is satisfied μ-almost surely, and if ν is a distribution such that (2.11) is satisfied ν-almost surely, then for each $n \in \mathbb{N}$,*

$$
\lim_{t \to \infty} \sup_{A \in \mathcal{F}_n} \left| \mu(A) - \int \mathbb{P}_f(u(t) \in A)\, \nu(df) \right| = 0. \tag{2.12}
$$

Furthermore, μ is translation invariant and satisfies, for each $p \in \mathbb{N}$,

$$
\int \sup_{x \in [-1,1]} |f(x)|^p\, \mu(df) < \infty. \tag{2.13}
$$

Proof To construct a translation invariant stationary distribution μ, we set $g = 0$ and apply Proposition 1.8.1, noting that (1.42) is implied by (2.3). Since g is translation invariant, so is μ. By the tightness estimates used in the proof of Proposition 1.8.1, we see that (2.13) holds. To show that (2.13) and translation invariance implies that (2.11) holds μ-almost surely, note that for $p \in \mathbb{N}$ and $\gamma > 0$,

$$
\begin{aligned}
\int \sup_{x \in \mathbb{R}} \left(\frac{|f(x)|}{1 + |x|^\gamma} \right)^p \mu(df) &\leq \sum_{n \in \mathbb{Z}} \int \sup_{x \in [n-1, n+1]} \left(\frac{|f(y)|}{1 + |n|^\gamma} \right)^p \mu(df) \\
&\leq \left(\sum_{n \in \mathbb{Z}} \frac{1}{1 + |n|^{p\gamma}} \right) \int \sup_{s \in [-1,1]} |f(x)|^p\, \mu(df)
\end{aligned}
$$

and take $p > \gamma^{-1}$. (2.12) is now immediate from Corollary 2.5.2. It also follows any two stationary distributions satisfying the hypotheses of the theorem agree on

$\mathcal{A}$ so, since $\mathcal{A}$ is a algebra, they also agree on $\sigma(\mathcal{A})$. It therefore suffices to show that $\sigma(\mathcal{A}) = \mathcal{B}(C_{tem})$.

We check that $\sigma(\mathcal{A})$ contains the open balls, which generate $\mathcal{B}(C_{tem})$. Fix $\varepsilon > 0$ and $\alpha \in C_{tem}$ and consider the open ball $B_\varepsilon(\alpha) \in \mathcal{B}(C_{tem})$.

$$\begin{aligned} B_\varepsilon(\alpha) &= \left\{ f \in C_{tem} : \sum_{\lambda \in 1/\mathbb{N}} 2^{-1/\lambda} \sup_{x \in \mathbb{R}} e^{-\lambda|x|} |f(x) - \alpha(x)| \wedge 1 < \varepsilon \right\} \\ &= \bigcup_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \left\{ f \in C_{tem} : \sum_{\lambda \in 1/\mathbb{N}} 2^{-1/\lambda} \times \right. \\ &\qquad \left. \sup_{x \in [-n,n]} e^{-\lambda|x|} |f(x) - \alpha(x)| \wedge 1 < \varepsilon - 1/m \right\}. \end{aligned}$$

It can be seen that

$$\left\{ f \in C([-n,n]) : \sum_{\lambda \in 1/\mathbb{N}} 2^{-1/\lambda} \sup_{x \in [-n,n]} e^{-\lambda|x|} |f(x) - \alpha(x)| \wedge 1 < \varepsilon - 1/m \right\}$$

is open in $C([-n,n])$ for each $n, m \in \mathbb{N}$ so it follows that

$$\left\{ f \in C_{tem} : \sum_{\lambda \in 1/\mathbb{N}} 2^{-1/\lambda} \sup_{x \in [-n,n]} e^{-\lambda|x|} |f(x) - \alpha(x)| \wedge 1 < \varepsilon - 1/m \right\} \in \mathcal{A}.$$

Hence $B_\varepsilon(\alpha) \in \sigma(\mathcal{A})$ and therefore $\sigma(\mathcal{A}) = \mathcal{B}(C_{tem})$ as required. $\square$

Chapter 3

Uniqueness of a living stationary distribution via a duality relation

3.1 Introduction

We consider nonnegative solutions of the SPDE

$$\partial_t u = \Delta u + \theta u - u^2 + \sqrt{2u}\, dW \qquad (3.1)$$

in dimension 1, where θ is a positive parameter and W is space-time white noise. Solutions of (3.1) arise as limits of a discrete contact process (Mueller and Tribe [18]). It is known (see Mueller and Tribe [17]) that solutions exhibit a phase transition: there exists some $\theta_c > 0$ such that from any compactly supported nonzero initial condition, when $\theta < \theta_c$, $u(t) = 0$ for large t with probability one and when $\theta > \theta_c$, there is a strictly positive probability that $u(t)$ is nonzero for all t. Uniqueness in law is also known for continuous, C^+_{tem}-valued solutions

(Tribe [31]).

We wish to show that, for $\theta > \theta_c$, there is a unique translation invariant stationary distribution satisfying a certain mixing property which is "living", i.e. gives zero probability to the zero configuration. Since the probability measure giving probability one to the zero configuration (the "everywhere dead" measure) is obviously a stationary distribution, we expect at least two stationary distributions and so the coupling technique doesn't apply. However, with the specific SPDE we are dealing with, we can prove that solutions to (3.1) satisfy a "self-dual" property. We use this to express the Laplace functional of u in terms of the death probability of an independent solution of (3.1). This leads to a proof of uniqueness of the living stationary distribution.

The result was inspired by a convergence theorem of Harris given as Theorem 3.3 on page 133 of Durrett [6], concerning a class of attractive particle systems ξ_t in continuous time on an integer lattice $\mathbb{Z}^d$. Here, uniqueness of a living stationary distribution is shown by proving that if $\mathbb{P}(\xi_0 = \emptyset) = 0$, then as $t \to \infty$, the probability of ξ_t having any sites alive in a set $B \subset \mathbb{Z}^d$ converges to the probability of the dual process starting from B living forever. This is a discrete space analogue of our result, and our interest was to try to implement ideas of duality in the SPDE setting.

3.2 Statement of result

Definition 3.2.1 *We call a probability measure μ on C_{tem}* pointwise mixing *if for each bounded Lipschitz function $h : \mathbb{R} \to \mathbb{R}$ and each $x \in \mathbb{R}$ we have*

$$\lim_{L\to\infty} \Big| \int h(\omega(x))h(\omega(x+L))\,\mu(d\omega) - \left(\int h(\omega(x))\,\mu(d\omega)\right)\left(\int h(\omega(x+L))\,\mu(d\omega)\right)\Big| = 0.$$

To compare with standard terminology, we note that μ is *mixing* if, for all $f, g \in L^2(C_{tem}, \mathcal{B}(C_{tem}), \mu)$, we have

$$\lim_{L\to\pm\infty} \int f(\omega(\cdot+L))g(\omega)\,\mu(d\omega) = \left(\int f(\omega)\,\mu(d\omega)\right)\left(\int f(\omega)\,\mu(d\omega)\right).$$

(See, for example, page 24 of Cornfeld [2].) By choosing $f(\omega) = g(\omega) = h(\omega(x))$ in the definitions above, it is clear that mixing implies pointwise mixing.

Theorem 3.2.2 *For $\theta > \theta_c$, there exists a unique translation invariant pointwise mixing stationary distribution μ on C^+_{tem} such that $\mu(\{f : f \neq 0\}) > 0$, and*

$$\int \sup_{x\in(-1,1)} |f(x)|\,\mu(df) < \infty. \tag{3.2}$$

Moreover, if u is a solution of (3.1) with initial condition $f \in C^+_{tem}$, and f is bounded below by some $\delta > 0$, then the distribution of $u(t)$ converges weakly to μ as $t \to \infty$.

Existence is shown in Section 3.5, uniqueness in Proposition 3.6.1 and the weak convergence result is proved in Proposition 3.6.4. In Lemma 3.6.3, we also show that if μ satisfies these hypotheses then $\mu(\{f : f \neq 0\}) = 1$.

3.3 Preliminary results

Let $\phi \in C_c^+(\mathbb{R})$ and suppose u is a C_{tem}-valued continuous solution of

$$\begin{aligned} \partial_t u &= \Delta u + \theta u + \sqrt{2u}\, dW \\ u(0) &= \phi. \end{aligned}$$

We state some well-known results about u which, by the comparison result given in the background chapter and uniqueness in distribution, we can also apply to solutions of (3.1).

Lemma 3.3.1 *There exists a constant $C > 0$ depending only on ϕ and θ such that for all $R \geq C$ and $t \geq 4$,*

$$\mathbb{P}\left(\int_0^t \int_R^\infty u(s,x)\, dx\, ds > 0\right) \leq Ce^{Ct}e^{-R^2/Ct}.$$

Proof Lemma 2.1 of Tribe [31] is a similar result in more generality with a $\sqrt{u}\, dW$ noise term. A scaling argument gives the result for our $\sqrt{2u}\, dW$ case. □

This gives us the following:

Corollary 3.3.2 *For all $T \geq 0$ there exists a constant $C = C(\phi, \theta, T)$ such that for all $x \in \mathbb{R}$*

$$\mathbb{P}\left(\sup_{s \leq T, y \in B_1(x)} |u(s,y)| > 0\right) \leq Ce^{-x^2/C}. \tag{3.3}$$

Furthermore, for all $p > 0$ and $T \geq 0$ there exists a constant $C = C(\phi, \theta, p, T)$ such that for all $x \in \mathbb{R}$,

$$\mathbb{E}\left[\sup_{s \leq T, y \in B_1(x)} |u(s,y)|^p\right] \leq Ce^{-x^2/C}. \tag{3.4}$$

Proof (3.3) is immediate from Lemma 3.3.1. To see (3.4), we apply Cauchy-Schwartz to get

$$\mathbb{E}\left[\sup_{s\le T, y\in B_1(x)} |u(s,y)|^p\right]$$
$$\le \mathbb{P}\left(\sup_{s\le T, y\in B_1(x)} |u(s,y)| > 0\right)^{1/2} \mathbb{E}\left[\sup_{s\le T, y\in B_1(x)} |u(s,x)|^{2p}\right]^{1/2}.$$

We can control the expectation here by (1.28). □

Since the bound in Lemma 3.3.1 converges to zero as $R \to \infty$ we see that with probability one, $u(t)$ is compactly supported for each t. It is also known, and a proof is given in Horridge [11], that there exists a constant $\alpha = \alpha(\theta)$ such that

$$\lim_{t\to\infty} \mathbb{P}\left(\operatorname{supp} u(t) \subseteq [-\alpha t, \alpha t]\right) = 1. \tag{3.5}$$

It is also well known (see Horridge [11]) that the distribution of the death time of u is given by

$$\mathbb{P}\left((u(t),1) = 0\right) = \exp\left(\frac{-\theta(\phi,1)}{1-e^{-\theta t}}\right). \tag{3.6}$$

We now consider the equation with the $-u^2$ term and use a different comparison to obtain superexponential moment bounds.

Lemma 3.3.3 *Let u be a C_{tem}-valued continuous solution of (3.1) with bounded initial condition $\phi \in C^+(\mathbb{R})$. Then for any $R > 0$ and $0 < \gamma < 2$ we have*

$$\sup_{t\ge 0, x\in\mathbb{R}} \mathbb{E}\left[\exp\left((u(t), \chi_{(x-R,x+R)})^\gamma\right)\right] < \infty.$$

Proof Given $x \in \mathbb{R}$ and $R > 0$, choose $\alpha_0 \in C_c^+(\mathbb{R})$ such that

$$\chi_{(x-R,x+R)} \le \alpha_0 \le \chi_{(x-R-1,x+R+1)}.$$

Then $\mathbb{E}\left[\exp\left((u(t),\chi_{(x-R,x+R)})^\gamma\right)\right] \leq \mathbb{E}[\exp\left((u(t),\alpha_0)\right)^\gamma]$. We first find a bound for $\mathbb{E}\left[e^{\lambda(u(t),\alpha_0)}\right]$ for each $\lambda > 0$ and use these to interpolate to the required superexponential bound. Let v be a C_{tem}-valued continuous solution of

$$\begin{aligned} \partial_t v &= \Delta v - 2\lambda v + \beta + \sqrt{2v}\, dW \\ v(0) &= \phi \end{aligned}$$

where $\beta = (\theta + 2\lambda)^2/4$. Completing the square shows that $-2\lambda u + \beta \geq \theta u - u^2$, so any moment bounds on v also apply to u by Proposition 1.5.3. Let ψ be the unique nonnegative solution of

$$\left.\begin{aligned} \partial_t \psi &= \Delta\psi - 2\lambda\psi + (\psi \wedge \lambda)\psi \\ \psi(0) &= \lambda\alpha_0. \end{aligned}\right\} \tag{3.7}$$

Then, for each $t \geq 0$ and $z \in \mathbb{R}$,

$$\begin{aligned} \psi(t,z) &= \lambda \int \alpha_0(y) G^\lambda(t,z,y)\, dy + \int_0^t \int (\psi \wedge \lambda - \lambda) G^\lambda(t-s,z,y)\, dy\, ds \\ &\leq \lambda e^{-\lambda t} \int \alpha_0(y) G(t,z,y)\, dy \end{aligned}$$

where $G^\lambda(s,z,y) = e^{-\lambda s} G(s,z,y)$. Hence

$$(\psi(t),1) \leq \lambda(\alpha_0,1)e^{-\lambda t} \leq 2\lambda(R+1)e^{-\lambda t}$$

and therefore

$$\int_0^t (\psi(s),1)\, ds \leq 2(R+1).$$

By comparing (3.7) with the same PDE with initial condition identically equal to λ, we see that for all $t \geq 0$ and $z \in \mathbb{R}$, $\psi(t,z) \in [0,\lambda]$, so ψ also solves

$$\partial_t \psi = \Delta\psi - 2\lambda\psi + \psi^2.$$

We fix $T > 0$, set $\xi(t) = \psi(T-t)$ for $0 \le t \le T$ and set

$$\mu(t) = \beta \int_t^T \int \xi(s,x)\,dx\,ds.$$

Then $\mu'(t) = -\beta(\xi(t), 1)$. Let $X_t = e^{(v(t),\xi(t))}e^{\mu(t)}$. By Ito's formula,

$$\begin{aligned}
dX_t &= X_t\left(\mu'(t) + (v(t), (\Delta\xi + \partial_t\xi - 2\lambda\xi + \xi^2)(t)) + \beta(\xi(t),1)\right)\,dt \\
&+ X_t \int \sqrt{2v(t,x)}\,\xi(t,x)\,W(dx\,dt) \\
&= X_t \int \sqrt{2v(t,x)}\,\xi(t,x)\,W(dx\,dt)
\end{aligned}$$

so X_t is a local martingale. We can therefore choose stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \to \infty$ almost surely and each $X_{t\wedge\tau_n}$ is a martingale. By Fatou's lemma we have

$$\mathbb{E}X_t \le \lim_{n\to\infty} \mathbb{E}X_{t\wedge\tau_n} = X(0),$$

hence

$$\begin{aligned}
\mathbb{E}\left[e^{(v(T),\lambda\alpha_0)}\right] &\le \exp\left((\phi,\psi(T)) + \beta\int_0^T (\psi(t),1)\,dt\right) \\
&\le \exp\left(2(R+1)(\lambda\sup\phi + \beta)\right)
\end{aligned}$$

which is bounded above by

$$\begin{aligned}
&\exp\left(2(R+1)(\lambda\sup\phi + \theta^2 + 4\lambda^2)\right) \\
&\qquad \le e^{2\theta^2(R+1)}\exp\left(2(R+1)(\lambda^2+1)(\sup\phi + 4)\right) \le Ce^{C\lambda^2}
\end{aligned}$$

for some constant $C = C(\theta,\phi,R)$. We now use these exponential moment bounds to get a superexponential bound. Let $X = (u(t),\alpha_0)$ and fix $\gamma \in (1,2)$. Then

$$\mathbb{E}\left[e^{X^\gamma}\right] \le \sum_{n=0}^{\infty} e^{(n+1)^\gamma}\mathbb{P}(X \ge n) \le \sum_{n=0}^{\infty} e^{(n+1)^\gamma}\mathbb{E}\left[e^{\lambda_n X}\right]e^{-\lambda_n n}$$

for any positive sequence λ_n. We set $\lambda_n = n^\delta$ for some $\delta \in (\gamma - 1, 1)$. It is then sufficient to show that $(n+1)^\gamma + Cn^{2\delta} - n^{\delta+1} \le -n$ for all sufficiently large n. Since the $n^{\delta+1}$ term dominates, this is clear. □

3.4 The duality relation

Proposition 3.4.1 *For $f \in C^+_{tem}$, let $\mathbb{P}_f$ denote the probability distribution of a solution of (3.1) with initial condition f. Then for all $f \in C^+_{tem}$, $\phi \in C^+_c(\mathbb{R})$ and $t \ge 0$, we have*

$$\mathbb{E}_f\left[e^{-(u(t),\phi)}\right] = \mathbb{E}_\phi\left[e^{-(u(t),f)}\right].$$

Proof Let u and v be independent C_{tem}-valued continuous solutions of (3.1) with respect to independent noises W_1 and W_2, with initial conditions f and ϕ respectively. For functions $\psi, \xi \in C_+(\mathbb{R})$ where the definition makes sense, set

$$g(\psi,\xi) = e^{-(\psi,\xi)}\left((\psi,\xi^2) + (\psi^2,\xi) - (\psi,\Delta\xi) - \theta(\psi,\xi)\right).$$

By Ito's formula and (1.2), we see that for $\xi \in C_c^\infty(\mathbb{R})_+$,

$$\begin{aligned} e^{-(u(t),\xi)} - \int_0^t g(u(r),\xi)\,dr &= -\int_0^t e^{-(u(s),\xi)} \int \sqrt{2u(s,x)}\,\xi(x)\,W_1(dx\,ds) \\ e^{-(v(t),\xi)} - \int_0^t g(v(r),\xi)\,dr &= -\int_0^t e^{-(v(s),\xi)} \int \sqrt{2v(s,x)}\,\xi(x)\,W_2(dx\,ds) \end{aligned}$$

which are easily seen to be true martingales. Also, if ψ and ξ are both C^2 and at least one of them is compactly supported, then we can use integration by parts to show that $g(\psi,\xi) = g(\xi,\psi)$. If we could extend the definition of g to all functions

$\psi, \xi \in C^+_{tem}$ and show that

$$e^{-(u(t),\xi)} - \int_0^t g(u(s),\xi)\,ds \quad \text{and} \quad e^{-(\psi,v(t))} - \int_0^t g(\psi, v(s))\,ds$$

were martingales for each $\xi, \psi \in C^+_{tem}$, we could apply Theorem 4.11 on page 192 of Ethier and Kurtz [8] to obtain the result immediately. Unfortunately, we can't make sense of $g(\psi, \xi)$ for non-differentiable ξ, so we follow the Ethier and Kurtz proof with smoothed-off approximations.

For $\varepsilon \in (0,1]$, choose $\rho_\varepsilon \in C_c^\infty(\mathbb{R})_+$ such that it is symmetric about 0, has support $[-\varepsilon, \varepsilon]$, and $\int \rho_\varepsilon = 1$. For $\psi \in C(\mathbb{R})_+$, set

$$\psi_\varepsilon(x) = \int \psi(y)\rho_\varepsilon(x-y)\,dy.$$

Let $F_\varepsilon(s,t) := \mathbb{E}\left[e^{-(u(s)_\varepsilon, v(t))}\right] = \mathbb{E}\left[e^{-(u(s),v(t)_\varepsilon)}\right]$. By the compact support property, for each $t \geq 0$ and $\varepsilon > 0$, $v(t)_\varepsilon$ is in $C_c^\infty(\mathbb{R})_+$ with probability one. Because u and v are independent and $\Theta_t\phi$ is the distribution of $v(t)$,

$$\begin{aligned} F_\varepsilon(s,t) &= \int \mathbb{E}\left[e^{-(u(s),\omega_\varepsilon)}\right] (\Theta_t\phi)(d\omega) \\ &= \int \left(e^{-(f,\omega_\varepsilon)} + \int_0^s \mathbb{E}[g(u(r),\omega_\varepsilon)]\,dr\right) (\Theta_t\phi)(d\omega). \end{aligned}$$

It is not difficult to show that $\int_0^s \mathbb{E}\left[|g(u(r), u(s)_\varepsilon)|\right] dr$ is finite, so

$$F_\varepsilon(s,t) = \mathbb{E}\left[e^{-(f,v(t))}\right] + \int_0^s \mathbb{E}[g(u(r), v(t)_\varepsilon)]\,dr$$

and hence, for $h > 0$,

$$F_\varepsilon(s+h,t) - F_\varepsilon(s,t) = \int_s^{s+h} \mathbb{E}[g(u(r), v(t)_\varepsilon)]\,dr. \tag{3.8}$$

Although $u(s)_\varepsilon$ isn't compactly supported in general, we can still get a similar formula for an increment in t by proceeding as follows. Choose a smooth function

α contained in $[0,1]$ which is 1 on $[-1,1]$ and supported on $[-2,2]$, and set $\alpha_n = \alpha(\cdot/n)$ for each $n \in \mathbb{N}$. Now

$$\mathbb{E}\left[e^{-(v(t),u(s)_\varepsilon\alpha_n)}\right] = \int_0^t \mathbb{E}[g(v(r),u(s)_\varepsilon\alpha_n)]\,dr.$$

Since $\sup_{n\in\mathbb{N},x\in\mathbb{R}}(|\alpha_n'(x)|+|\alpha_n''(x)|)<\infty$, checking dominated convergence as $n\to\infty$ gives

$$F_\varepsilon(s,t+h) - F_\varepsilon(s,t) = \int_t^{t+h} \mathbb{E}[g(v(r),u(s)_\varepsilon)]\,dr. \tag{3.9}$$

Taking limits as $h\to 0$ in (3.8) and (3.9) gives

$$\begin{aligned} \frac{\partial}{\partial s}F_\varepsilon(s,t) &= \mathbb{E}[g(u(s),v(t)_\varepsilon)] \\ \frac{\partial}{\partial t}F_\varepsilon(s,t) &= \mathbb{E}[g(v(t),u(s)_\varepsilon)]\,. \end{aligned}$$

By Lemma 4.10 on page 192 of Ethier and Kurtz [8], we have

$$F_\varepsilon(t,0) - F_\varepsilon(0,t) = \int_0^t \left(\frac{\partial}{\partial s}F_\varepsilon(s,t-s) - \frac{\partial}{\partial t}F_\varepsilon(s,t-s)\right) ds.$$

The left hand side of this converges by the dominated convergence theorem to $\mathbb{E}_f\left[e^{-(u(t),\phi)}\right] - \mathbb{E}_\phi\left[e^{-(u(t),f)}\right]$ as $\varepsilon\to 0$, so it suffices to check that the right hand side converges to zero. We first note that

$$(\Delta(u(s)_\varepsilon), v(t-s)) = (u(s), \Delta(v(t-s)_\varepsilon))$$

almost surely, so

$$\begin{aligned} |F_\varepsilon(t,0) - F_\varepsilon(0,t)| \;\le\; & \int_0^t \mathbb{E}\left|\left(u(s),(v(t-s)_\varepsilon)^2 - v(t-s)^2\right)\right| ds \,+ \\ & \int_0^t \mathbb{E}\left|\left(u(s)-u(s)_\varepsilon, v(t-s)^2\right)\right| ds \,+ \\ & \int_0^t \mathbb{E}\left|\left(u(s)^2, v(t-s)_\varepsilon - v(t-s)\right)\right| ds \,+ \\ & \int_0^t \mathbb{E}\left|\left(u(s)^2 - (u(s)_\varepsilon)^2, v(t-s)\right)\right| ds. \end{aligned}$$

Clearly, $v(t-s)_\varepsilon(x)$ converges to $v(t-s,x)$ almost surely as $\varepsilon \to 0$ so to show that the first term converges to zero, it suffices to check dominated convergence, i.e.

$$\int_0^t \int \mathbb{E}\left[u(s,x)\left(\sup_{0<\varepsilon\le 1}\left|v(t-s)_\varepsilon(x)^2 - v(t-s,x)^2\right|\right)\right] dx\,ds < \infty. \quad (3.10)$$

Now

$$\begin{aligned}\sup_{0<\varepsilon\le 1}\left|v(s)_\varepsilon(x)^2 - v(s,x)^2\right| &\le \sup_{0<\varepsilon\le 1}\left(\left|v(s)_\varepsilon(x) + v(s,x)\right|^2\right) \\ &\le 4\left(\sup_{y\in B_1(x)} v(s,y)^2\right)\end{aligned}$$

so by (3.4) and the independence of u and v, we see that for some C depending only on T, ϕ and θ,

$$\begin{aligned}&\mathbb{E}\left[u(s,x)\left(\sup_{0<\varepsilon\le 1}\left|v(t-s)_\varepsilon(x)^2 - v(t-s,x)^2\right|\right)\right] \\ &\quad\le 4\mathbb{E}[u(s,x)]\,\mathbb{E}\left[\sup_{y\in B_1(x)} v(s,y)^2\right] \le C\mathbb{E}\left[u(s,x)e^{-x^2/C}\right]\end{aligned}$$

so (3.10) holds by Lemma 1.1.3 and Proposition 1.5.3. The other terms follow a similar pattern. □

3.5 Existence

In this section we construct a nontrivial, translation invariant measure on C^+_{tem} which is both pointwise mixing and stationary with respect to (3.1). We use a different method of constructing a stationary distribution to that described in Section 1.8 since it is not necessarily true that a Cesaro average of pointwise mixing measures is pointwise mixing. For example, it is possible to construct families $\{X_s\}_{s\in[0,1]}$ and $\{Y_s\}_{s\in[0,1]}$ of real-valued random variables such that X_s and Y_s are independent for each $s \in [0,1]$, but $\int_0^1 X_s\, ds$ and $\int_0^1 Y_s\, ds$ are not.

Intuitively, we start a solution of (3.1) from an infinite initial condition and show it is stochastically decreasing in time. A similar idea is used for a discrete site particle system in Theorem 2.7 of Durrett [6], where the author starts with every site occupied and passes to a limit as $t \to \infty$.

Although the duality formula is used in Lemma 3.5.6 below, it ought to be possible to get round this and use the method to construct a pointwise mixing stationary measure for a more general SPDE as long as we have a $-u^\gamma$ term, for some $\gamma > 1$, to pull down a solution started from an infinite initial condition in finite time.

Lemma 3.5.1 *Suppose $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of translation invariant, pointwise mixing distributions on C_{tem} which converge weakly to some distribution μ. Then μ is also pointwise mixing.*

Proof Since μ is translation invariant, we only need to check the definition for $x = 0$. By Skorokhod's Theorem, there exists a probability space with C_{tem}-valued random variables $\{u_n\}_{n\in\mathbb{N}}$ and u such that each u_n has distribution μ_n, u has distribution μ and u_n converges almost surely to u. Now, for $h : \mathbb{R} \to \mathbb{R}$ continuous and bounded,

$$\begin{aligned}&\left|\int h(\omega(0))h(\omega(L))\,\mu(d\omega) - \left(\int h(\omega(0))\,\mu(d\omega)\right)^2\right| \\ &\quad = \left|\mathbb{E}[h(u(0))h(u(L))] - \mathbb{E}[h(u(0))]^2\right|\end{aligned}$$

$$
\begin{aligned}
&= \Big| \mathbb{E}[h(u(0))\,(h(u(L)) - h(u_n(L))) + h(u_n(L))\,(h(u(0)) - h(u_n(0)))] \\
&+ \mathbb{E}[h(u_n(0))h(u_n(L))] - \mathbb{E}[h(u_n(0))]^2 \\
&+ \mathbb{E}[h(u_n(0)) + h(u(0))]\,\mathbb{E}[h(u_n(0)) - h(u(0))] \Big| \\
&\leq \left(4 \sup_{z\in\mathbb{R}} |h(z)|\right) \mathbb{E}[|h(u_n(0)) - h(u(0))|] \\
&+ \left|\mathbb{E}[h(u_n(0))h(u_n(L))] - \mathbb{E}[h(u_n(0))]^2\right|.
\end{aligned}
$$

Choosing n suitably large and then L suitably large, we see that this can be made arbitrarily small, as required. □

We need the following estimate on the tails of $G(t, y)$ and $G(t, y)^2$.

Lemma 3.5.2 *For all $X \geq 1$,*

$$
\begin{aligned}
\int_{|y|\geq X} G(t, y)\, dy &\leq t^{1/2} e^{-X^2/4t}, \\
\int_{|y|\geq X} G(t, y)^2\, dy &\leq e^{-X^2/4t}.
\end{aligned}
$$

Proof Rescaling Theorem 1.4 on page 7 of Durrett [7] gives

$$
\int_X^\infty e^{-y^2/4t}\, dy \leq \frac{2te^{-X^2/4t}}{X}.
$$

The result easily follows. □

Here is our main lemma for checking pointwise mixing. Once this has been established in the Lipschitz case for a finite initial condition at a finite time, we will be able to use Lemma 3.5.1 to pass to the required limits.

Lemma 3.5.3 *Suppose u is a continuous, C_{tem}-valued solution of (1.1) with a, b Lipschitz, and $u(0) = N$ is a deterministic constant. Then for each $t \geq 0$, the distribution of $u(t)$ is pointwise mixing.*

Proof Fix $T > 0$. For each $L \geq (8T)^{1/2} \vee 4$, we follow Walsh [32] in constructing a solution u^L to

$$\begin{aligned}
\partial_t u^L &= \Delta u^L + a(u^L) + b(u^L)\, dW \text{ for } t > 0, x \in (-L/2, L/2), \\
\partial_x u^L(t, \pm L/2) &= 0 \text{ for } t > 0, \\
u(0, x) &= N \text{ for } x \in [-L/2, L/2]
\end{aligned}$$

on $\mathbb{R}_+ \times [-L/2, L/2]$, where W is the same white noise as that for u. Then u^L solves the integral equation

$$\begin{aligned}
u^L(t,x) &= N + \int_0^t \int_{-L/2}^{L/2} a(u(s,y)) G^L(t-s,x,y)\, dy\, ds \\
&+ \int_0^t \int_{-L/2}^{L/2} b(u(s,y)) G^L(t-s,x,y)\, W(dy\, ds)
\end{aligned}$$

where

$$G^L(t,x,y) = \frac{1}{\sqrt{4\pi t}} \sum_{n \in \mathbb{Z}} \left(e^{-(y-x-2nL)^2/4t} + e^{-(y+x-L(2n+1))^2/4t} \right).$$

Note that since Walsh defines a solution on $[0, L]$, we have to shift the Green's function given on page 312 of [32]. Now for some constant $C = C(a, b, T)$, we have, for $t \leq T$, $\lambda > 0$ and $x \in [-L/2, L/2]$,

$$\begin{aligned}
&\mathbb{E}\left[|u(t,x) - u^L(t,x)|^2 e^{-\lambda |x|} \right] \\
&\leq C e^{-\lambda |x|} \int_0^t \int_{-L/2}^{L/2} \mathbb{E}\left[|u(s,y) - u^L(s,y)|^2 \right] G(t-s,x,y)\, dy\, ds \\
&+ C e^{-\lambda |x|} \mathbb{E}\left[\left(\int_0^t \int_{-L/2}^{L/2} (|u^L(s,y)| + 1)(G(t-s,x,y) - \right.\right. \\
&\qquad\qquad \left.\left. G^L(t-s,x,y))\, dy\, ds \right)^2 \right]
\end{aligned}$$

$$
\begin{aligned}
&+ \quad Ce^{-\lambda|x|} \int_0^t \int_{|y|\geq L/2} \mathbb{E}\left[|u(s,y)|^2+1\right] G(t-s,x,y)\,dy\,ds \\
&+ \quad Ce^{-\lambda|x|} \int_0^t \int_{-L/2}^{L/2} \mathbb{E}\left[|u(s,y)-u^L(s,y)|^2\right] G(t-s,x,y)^2\,dy\,ds \\
&+ \quad Ce^{-\lambda|x|} \int_0^t \int_{-L/2}^{L/2} \mathbb{E}\left[u^L(s,y)^2+1\right] (G(t-s,x,y) - \\
&\qquad\qquad\qquad\qquad\qquad G^L(t-s,x,y))^2\,dy\,ds \\
&+ \quad Ce^{-\lambda|x|} \int_0^t \int_{|y|\geq L/2} \mathbb{E}\left[u(s,y)^2+1\right] G(t-s,x,y)^2\,dy\,ds.
\end{aligned}
$$

We label the terms on the right by $J_1, \ldots, J_6$. Note that for $x \in [-L/4, L/4]$, $t \leq T$,

$$
\begin{aligned}
&\int_{-L/2}^{L/2} \left(G^L(t,x,y) - G(t,x,y)\right)\,dy = 1 - \int_{-L/2}^{L/2} G(t,x,y)\,dy \\
&= \int_{|y|\geq L/2} G(t,x,y)\,dy \leq \int_{|y|\geq L/4} G(t,y)\,dy \leq T^{1/2} e^{-L^2/64T}.
\end{aligned}
$$

So

$$
\begin{aligned}
J_2 + J_3 + J_6 \quad \leq \quad & Ce^{-L^2/64T} \Bigg(\sup_{s\leq T, y\in\mathbb{R}} \mathbb{E}\left[u(s,y)^2\right] + \\
& \sup_{s\leq T, y\in[-L/2,L/2]} \mathbb{E}\left[u^L(s,y)^2\right] + 1 \Bigg)
\end{aligned}
$$

for some constant $C = C(a,b,T)$. Applying (1.8) we see that, for some $C = C(a,b,\lambda,T)$,

$$
\begin{aligned}
J_1 \quad &\leq \quad C \int_0^t \int_{-L/2}^{L/2} \mathbb{E}\left[|u(s,y)-u^L(s,y)|^2 e^{-\lambda|y|}\right] G(2(t-s),x,y)\,dy\,ds \\
&\leq \quad C \int_0^t \sup_{|y|\leq L/4} \mathbb{E}\left[|u(s,y)-u^L(s,y)|^2 e^{-\lambda|y|}\right]\,ds \\
&+ \quad Ce^{-\lambda L/8} \sup_{s\leq T, |y|\in[L/4,L/2]} \mathbb{E}\left[\left(u(s,y)^2 + u^L(s,y)^2\right) e^{-\lambda|y|/2}\right].
\end{aligned}
$$

Similarly,

$$\begin{aligned} J_4 &\leq C \int_0^t \frac{1}{\sqrt{t-s}} \sup_{|y| \leq L/4} \mathbb{E}\left[|u(s,y) - u^L(s,y)|^2 e^{-\lambda|y|}\right] ds \\ &+ Ce^{-\lambda L/8} \sup_{s \leq T, |y| \in [L/4, L/2]} \mathbb{E}\left[\left(u(s,y)^2 + u^L(s,y)^2\right) e^{-\lambda|y|/2}\right]. \end{aligned}$$

By Lemma 3.5.4 below, for some $C = C(T)$,

$$J_5 \leq Ce^{-L^2/64T} \sup_{s \leq T, y \in [-L/2, L/2]} \mathbb{E}\left[u^L(s,y)^2 + 1\right].$$

By a standard Green's function argument, and using Lemma 3.5.4, we see that

$$\sup_{s \leq T} \sup_{y \in [-L/4, L/4]} \mathbb{E}\left[u^L(s,y)^2\right]$$

is bounded independently of L for $L \geq (8T)^{1/2} \vee 4$. So

$$\begin{aligned} &\sup_{y \in [-L/4, L/4]} \mathbb{E}\left[|u(t,y) - u^L(t,y)|^2 e^{-\lambda|y|}\right] \\ &\quad \leq \int_0^t \frac{C}{\sqrt{t-s}} \sup_{y \in [-L/4, L/4]} \mathbb{E}\left[|u(s,y) - u^L(s,y)|^2 e^{-\lambda|y|}\right] ds + H(L) \end{aligned}$$

where $H(L)$ converges to zero as $L \to \infty$ and C is independent of L. Applying Gronwall gives

$$\lim_{L \to \infty} \sup_{y \in [-L/4, L/4]} \mathbb{E}\left[|u(t,y) - u^L(t,y)|^2 e^{-\lambda|y|}\right] = 0$$

and in particular, $\lim_{L \to \infty} \mathbb{E}\left[|u(t,0) - u^L(t,0)|^2\right] = 0$. Now let u_2^L be a solution defined analogously on the space interval $[L/2, 3L/2]$. Then u^L and u_2^L are independent, as the solutions depend on disjoint regions of the white noise W. Also, $|u(t,L) - u_2^L(t,L)|$ is equal in distribution to $|u(t,0) - u^L(t,0)|$. If $h : \mathbb{R} \to \mathbb{R}$ is Lipschitz and bounded,

$$\begin{aligned} &|\mathbb{E}[h(u(t,0))h(u(t,L))] - \mathbb{E}[h(u(t,0))]\,\mathbb{E}[h(u(t,L))]| \\ &\quad \leq 2\,(\sup|h|)\left(\mathbb{E}\left[|h(u^L(t,0)) - h(u(t,0))|\right] + \mathbb{E}\left[|h(u_2^L(t,L)) - h(u(L))|\right]\right) \end{aligned}$$

which converges to zero as $L \to \infty$. $\square$

Lemma 3.5.4 *There exists a constant C such that for all $T > 0$, $L \geq (8T)^{1/2} \vee 4$, $x \in [-L/2, L/2]$ and $s \leq T$,*

$$\int_{-L/2}^{L/2} G^L(s,x,y)^2 \, dy \, ds \leq C \left(s^{-1/2} + e^{-L^2/4T} \right),$$

and such that if $x \in [-L/4, L/4]$, $s \leq T$,

$$\int_{-L/2}^{L/2} \left(G^L(s,x,y) - G(s,x,y) \right)^2 \, dy \leq C e^{-L^2/64T}.$$

Proof

$$\begin{aligned} G^L(s,x,y) &\leq G(s,y,x) + G(s,y-L,-x) + G(s,y+L,x) \\ &+ \sum_{|n| \geq 2} \left(G(s,y+nL,x) + G(s,y+nL,-x) \right). \end{aligned}$$

For $L \geq (8T)^{1/2}$ and $s \leq T$,

$$\begin{aligned} &\int_{-L/2}^{L/2} \left(\sum_{|n| \geq 2} G(s,y+nL,x) \right)^2 dy \\ &= \sum_{|m|,|n| \geq 2} \int_{-L/2}^{L/2} G(s,y+mL,x) G(s,y+nL,x) \, dy \\ &\leq \left(\sum_{|n| \geq 2} \left(\int_{-L/2}^{L/2} G(s,y+nL,x)^2 \, dy \right)^{1/2} \right)^2 \\ &\leq \left(\sum_{n \in \mathbb{Z} \setminus 0} \left(\int_{|n|L}^{\infty} G(s,y)^2 \, dy \right)^{1/2} \right)^2 \\ &\leq \left(\sum_{n \in \mathbb{Z} \setminus 0} e^{-|n| L^2/8s} \right)^2 \leq \frac{4e^{-L^2/4T}}{\left(1 - e^{-L^2/8T}\right)^2} \leq 11 e^{-L^2/4T} \end{aligned}$$

since $4(1-e^{-1})^{-2} \leq 11$. So for some constant C,

$$\begin{aligned}\int_{-L/2}^{L/2} G^L(s,x,y)^2\,dy\,ds &\leq C\left(\int G(s,y)^2\,dy + e^{-L^2/4T}\right)\\ &\leq C\left(s^{-1/2} + e^{-L^2/4T}\right).\end{aligned}$$

If $|x| \leq L/4$,

$$\int_{-L/2}^{L/2} G(s,y+L,x)^2\,dy \leq \int_{-L/4}^{\infty} G(s,y)^2\,dy \leq e^{-L^2/64T}$$

and similarly for $\int_{-L/2}^{L/2} G(s,y-L,x)^2\,dy$, so

$$\int \left(G^L(s,x,y) - G(s,x,y)\right)^2\,dy \leq Ce^{-L^2/64T}$$

for some constant C. □

Lemma 3.5.5 *Lemma 3.5.3 also applies for solutions of (3.1).*

Proof We recall that in Chapter 1 we constructed the distribution of a solution of (3.1) as a weak limit of Lipschitz approximations, and apply Lemma 3.5.1. □

For each $f \in C^+_{tem}$ and $t > 0$, we set

$$\mu_t^f = \mathbb{P}_f(u(t) \in \cdot),$$

where u is a solution of (3.1). In Lemma 3.4 of Tribe [31] it is checked that $\{\mu_t^f\}_{f\in C^+_{tem}}$ is tight for each $t > 0$. We can also show tightness of $\{\mu_t^f\}_{t\geq 1, f\in C^+_{tem}}$ by applying the Markov property at $t = 1/2$ and using a Green's function estimate to show that solutions stay well-behaved after $t = 1$.

For each $t \geq 1$, there exists a sequence of positive reals $N_n(t) \nearrow \infty$ such that $\mu_t^{N_n^{(t)}}$ converges to a weak limit which we call μ_t^∞. Furthermore, there exists a sequence $t_n \nearrow \infty$ such that $\mu_{t_n}^\infty$ converges weakly to a measure μ_∞^∞.

It follows from Lemma 3.5.1 that μ_∞^∞ is pointwise mixing. We claim that μ_∞^∞ is nontrivial — this turns out to be an easy corollary of the main result, so we defer proof until Corollary 3.6.5. It therefore remains to show that μ_∞^∞ is stationary.

Let $\mathcal{H}_0$ be the class of functions $H : C_{tem}^+ \to \mathbb{R}$ given by $H(f) = e^{-(\phi,f)}$ for some $\phi \in C_c^+(\mathbb{R})$.

Lemma 3.5.6 *Each function $H \in \mathcal{H}_0$ is nonincreasing and*

$$f \mapsto \int H(\omega)\mathbb{P}_f(u(t) \in d\omega) \tag{3.11}$$

is continuous. Suppose $\mathcal{H}$ is a set of bounded measurable functions $H : C_{tem}^+ \to \mathbb{R}$ which is closed under addition, scalar multiplication and bounded pointwise convergence, and contains $\mathcal{H}_0$. Then $\mathcal{H}$ contains all bounded measurable functions.

Proof Choose $H \in \mathcal{H}_0$. Then H is clearly nonincreasing. By the duality result, for any $f, g \in C_{tem}^+$ and $\lambda > 0$,

$$\begin{aligned}
&\left|\mathbb{E}_f\left[e^{-(u(t),\phi)}\right] - \mathbb{E}_g\left[e^{-(u(t),\phi)}\right]\right| \\
&\quad = \left|\mathbb{E}_\phi\left[e^{-(u(t),f)} - e^{-(u(t),g)}\right]\right| \leq \mathbb{E}_\phi\left[(u(t), |f-g|)\right] \\
&\quad \leq \left(\sup_{y\in\mathbb{R}} |f(y) - g(y)|e^{-\lambda|y|}\right) \int e^{\lambda|y|}\mathbb{E}_\phi\left[u(t,y)\right] dy.
\end{aligned}$$

The integral here is finite by (3.4). Hence (3.11) is continuous. Clearly, $\mathcal{H}_0$ is closed under multiplication, so by Corollary 4.4 on page 497 of Ethier and Kurtz [8], $\mathcal{H}$ contains all bounded $\sigma(\mathcal{H}_0)$-measurable functions. By choosing a sequence of functions ϕ_n approximating a point mass δ_x, we see that for each $x \in \mathbb{R}$, $0 \le a \le b$, the set

$$\{f \in C^+_{tem} : f(x) \in [a, b]\}$$

is $\sigma(\mathcal{H}_0)$-measurable, and by taking countable intersections of these, we can generate the closed balls of C^+_{tem}. It therefore follows that $\mathcal{H}_0$ generates the whole of $\mathcal{B}(C^+_{tem})$. □

Lemma 3.5.7 *For each $T \ge 0$ and $t > 0$, $\Theta_T \mu_t^\infty = \mu_{T+t}^\infty$*

Proof We already know that $\Theta_T \mu_t^N = \mu_{T+t}^N$ for each $N > 0$. Choose any $H \in \mathcal{H}_0$. Then

$$\begin{aligned}\int H(\omega)\Theta_T\mu_t^\infty(d\omega) &= \int \left(\int H(\omega)\mathbb{P}_f(u(T) \in d\omega) \right) \mu_t^\infty(df) \\ &= \lim_{N_n^{(t)} \to \infty} \int \left(\int H(\omega)\mathbb{P}_f(u(T) \in d\omega) \right) \mu_t^{N_n^{(t)}}(df) \\ &= \lim_{N_n^{(t)} \to \infty} \int H(\omega)\, \Theta_T \mu_t^{N_n^{(t)}}(d\omega) = \lim_{N_n^{(t)} \to \infty} \int H(\omega)\, \mu_{T+t}^{N_n^{(t)}}(d\omega).\end{aligned}$$

$N_n^{(t)}$ isn't necessarily the same subsequence as $N_n^{(T+t)}$, but since H is nonincreasing, by Proposition 1.5.2 we can see that $\int H(\omega)\, \mu_{T+t}^N(d\omega)$ is nonincreasing in N. Hence the rightmost term converges to $\int H(\omega)\, \mu_{T+t}^\infty(d\omega)$. The class of H such that

$$\int H(\omega)\, \Theta_T \mu_t^\infty(d\omega) = \int H(\omega)\, \mu_{T+t}^\infty(d\omega) \tag{3.12}$$

is closed under bounded pointwise convergence by the dominated convergence theorem, so by Lemma 3.5.6, we see that (3.12) holds for all bounded measurable H. Hence $\Theta_t \mu_t^\infty = \mu_{T+t}^\infty$. □

We now show that the family of measures $\{\mu_t\}_{t\geq 1}$ is stochastically decreasing in t.

Lemma 3.5.8 *If $H : C^+_{tem} \to \mathbb{R}$ is continuous, bounded and nondecreasing then*

$$\int H(\omega)\, \mu_t^\infty(d\omega) \searrow \int H(\omega)\, \mu_\infty^\infty(d\omega)$$

as $t \to \infty$.

Proof Set $\alpha(x) = 1 + |x|$. By tightness, for each $t > 0$, $\{\mu_t^{N\alpha}\}_{N>0}$ converges along a subsequence to a weak limit $\tilde{\mu}_t^\infty$. Now for bounded continuous nondecreasing H, $0 < s < t$ and $M, N > 0$,

$$\begin{aligned}
&\int H(\omega)\, \mu_t^N(d\omega) \\
&\quad = \int_{\{f\leq M\alpha\}} \int H(\omega)\, \mathbb{P}_f(u(s) \in d\omega)\, \mu_{t-s}^N(df) \\
&\quad + \int_{\{f\not\leq M\alpha\}} \int H(\omega)\, \mathbb{P}_f(u(s) \in d\omega)\, \mu_{t-s}^N(df) \\
&\quad \leq \int H(\omega)\, \mathbb{P}_{M\alpha}(u(s) \in d\omega) + \left(\sup_{z\in C^+_{tem}} |H(z)|\right) \mu_{t-s}^N(\{f \not\leq M\alpha\}) \\
&\quad \leq \int H(\omega)\, \tilde{\mu}_s^\infty(d\omega) + \left(\sup_{z\in C^+_{tem}} |H(z)|\right) \mu_{t-s}^N(\{f \not\leq M\alpha\}),
\end{aligned} \tag{3.13}$$

where the last two inequalities follow from Proposition 1.5.2. We now show that for each $N > 0$, $\mu_{t-s}^N(\{f \not\leq M\alpha\})$ converges to zero as $M \to \infty$. By the moment bounds in Section 1.3 it is easy to show that for each $x \in \mathbb{R}$ and $p > 0$,

$$\int \sup_{y\in B_1(x)} |f(y)|^p\, \mu_{t-s}^N(df) < \infty$$

and by translation invariance, this is independent of x. Now

$$\begin{aligned}\int \sup_{y\in\mathbb{R}} \left(\frac{|f(y)|}{1+|y|}\right)^p \mu_{t-s}^N(df) &\leq \sum_{n\in\mathbb{Z}} \int \sup_{y\in(n-1,n+1)} \left(\frac{|f(y)|}{1+|n|}\right)^p \mu_{t-s}^N(df) \\ &\leq \left(\int \sup_{y\in(-1,1)} |f(y)|^p \mu_{t-s}^N(df)\right) \sum_{n\in\mathbb{Z}} \frac{1}{(1+|n|)^p}\end{aligned}$$

which converges for $p > 1$. Hence $\sup_{y\in\mathbb{R}} |f(y)|/\alpha(y)$ is finite μ_{t-s}^N-almost surely and so $\mu_{t-s}^N(\{f \not\leq M\alpha\})$ converges to zero as $M \to \infty$. This gives, from (3.13),

$$\int H(\omega)\, \mu_t^\infty(d\omega) \leq \int H(\omega)\, \tilde{\mu}_s^\infty(d\omega). \tag{3.14}$$

We now show that $\tilde{\mu}_s^\infty = \mu_s^\infty$, since then the result follows immediately from (3.14). Suppose $J \in \mathcal{H}_0$. Since J is nonincreasing, by Proposition 1.5.2

$$\int J(\omega)\, \mu_s^\infty(d\omega) \geq \int J(\omega)\, \tilde{\mu}_s^\infty(d\omega),$$

and

$$\int J(\omega)\, \mu_s^M(d\omega) \leq \int J(\omega)\, \mu_s^{M\wedge N\alpha}(d\omega) \tag{3.15}$$

for each $M > 0$. Since $M \wedge N\alpha$ converges to $N\alpha$ as $M \to \infty$, by continuity of (3.11) we have

$$\lim_{M\to\infty} \int J(\omega)\, \mu_s^{M\wedge N\alpha}(d\omega) = \int J(\omega)\, \mu_s^{N\alpha}(d\omega).$$

so, letting $M \to \infty$ and then $N \to \infty$ in (3.15) we obtain

$$\int J(\omega)\, \mu_s^\infty(d\omega) \leq \int J(\omega)\, \tilde{\mu}_s^\infty(d\omega).$$

By Lemma 3.5.6, equality holds for all bounded measurable J so $\tilde{\mu}_s^\infty = \mu_s^\infty$. □

Proposition 3.5.9 *μ_∞^∞ is nontrivial, translation invariant, pointwise mixing and stationary, and*

$$\int \sup_{x\in(-1,1)} |\omega(x)|\, \mu_\infty^\infty(d\omega) < \infty. \tag{3.16}$$

Proof Since μ_t^∞ is stochastically decreasing, it suffices to check (3.16) for μ_1^∞. This follows from the moment bounds in Tribe [31] used to check tightness. Suppose $H : C_{tem}^+ \to \mathbb{R}$ is bounded and measurable and such that (3.11) is continuous. Then by Lemma 3.5.7 and Lemma 3.5.8

$$\begin{aligned}
\int H(\omega)\, \Theta_T \mu_\infty^\infty(d\omega) &= \int \left(\int H(\omega)\, \mathbb{P}_f(u(T) \in d\omega) \right) \mu_\infty^\infty(df) \\
&= \lim_{t_n\to\infty} \int \left(\int H(\omega)\, \mathbb{P}_f(u(T) \in d\omega) \right) \mu_{t_n}^\infty(df) \\
&= \lim_{t_n\to\infty} \int H(\omega)\, \mu_{t_n+T}^\infty(d\omega) = \int H(\omega)\, \mu_\infty^\infty(d\omega).
\end{aligned}$$

Applying Lemma 3.5.6 again completes the proof. □

3.6 Uniqueness

We define the death time τ^ϕ for $\phi \in C_c^+(\mathbb{R})$ as

$$\tau^\phi := \inf\{t > 0 : (u(t), 1) = 0\}$$

where u is a solution of (3.1) with $u(0) = \phi$. By uniqueness, the distribution of τ^ϕ is uniquely determined.

Proposition 3.6.1 *Let $\theta > \theta_c$. Suppose μ is a translation invariant, pointwise mixing stationary distribution on C_{tem}^+ such that*

$$\int \sup_{x\in(-1,1)} f(x)\, \mu(df) < \infty$$

and $\int f(0)\,\mu(df) > 0$. *Choose any* $\phi \in C_c^+(\mathbb{R})$. *Then*

$$\int e^{-(f,\phi)}\,\mu(df) = \mathbb{P}\left(\tau^\phi < \infty\right).$$

In particular, the stationary distribution satisfying these hypotheses is unique.

Proof Let v be a solution of (3.1) with initial condition ϕ, and let u_0 have distribution μ and be independent of v. Then by stationarity and our duality relation, for each $t > 0$,

$$\int e^{-(f,\phi)}\,\mu(df) = \mathbb{E}\left[e^{-(u_0,v(t))}\right].$$

Since $\mathbb{E}\left[e^{-(u_0,v(t))}\right] \geq \mathbb{P}\left(\tau^\phi < t\right)$, we have the lower bound immediately by letting $t \to \infty$. Also, for each $R > 0$,

$$\mathbb{E}\left[e^{-(u_0,v(t))}\right] \leq \mathbb{P}\left((u_0, v(t)) < R\right) + e^{-R}.$$

Since $\theta > \theta_c$, $\mathbb{P}\left(\tau^\phi = \infty\right) > 0$. Now

$$\mathbb{P}\left((u_0, v(t)) \geq R\right) = \mathbb{P}\left((u_0, v(t)) \geq R \mid \tau^\phi \geq t\right)\mathbb{P}\left(\tau^\phi \geq t\right).$$

so it suffices to show that there is a subsequence $t_n \nearrow \infty$ such that for each fixed $R > 0$,

$$\mathbb{P}\left((u_0, v(t_n)) \geq R \mid \tau^\phi \geq t_n\right) \to 1. \tag{3.17}$$

By Lemma 3.6.2 below, there exists a subsequence $t_n \nearrow \infty$ and some constant $c = c(\theta)$ such that

$$\mathbb{P}\left((v(t_n), 1) < c\log t_n \mid \tau^\phi \geq t_n\right) \leq \frac{\mathbb{P}\left(0 < (v(t_n), 1) < c\log t_n\right)}{\mathbb{P}\left(\tau^\phi = \infty\right)} \to 0$$

and hence

$$\mathbb{P}\left((v(t_n), 1) \geq c\log t_n \mid \tau^\phi \geq t_n\right) \to 1. \tag{3.18}$$

By Lemma 3.6.3 below,

$$\mathbb{P}\left((u_0, v(t_n)) \geq R \mid (v(t_n), 1) \geq c \log t_n\right) \to 1. \tag{3.19}$$

In other words, the total mass of v conditional on it not dying out grows in t, and if the total mass of v is growing quickly enough, there is a high probability that enough of it will intersect with u_0. From (3.18) and (3.19) we can deduce (3.17), and the result follows. □

Lemma 3.6.2 *Let v be a solution of (3.1) with initial condition $\phi \in C_c^+(\mathbb{R})$. For each $L > 0$,*

$$\lim_{t \to \infty} \mathbb{P}\left(0 < (v(t), 1) \leq L\right) = 0. \tag{3.20}$$

Furthermore, there exists a subsequence $t_n \nearrow \infty$ and a constant $c = c(\theta)$ such that

$$\lim_{n \to \infty} \mathbb{P}\left(0 < (v(t_n), 1) \leq c \log t_n\right) = 0. \tag{3.21}$$

Proof Following the argument in the proof of Theorem 3.3 in Durrett [6], we note that for each $t \geq 0$ and $L > 0$,

$$\begin{aligned}
\mathbb{P}\left(\tau^{\phi} \in (t, t+1]\right) &= \mathbb{P}\left(v(t) \neq 0 \text{ and } v(t+1) = 0\right) \\
&\geq \mathbb{P}\left(0 < (v(t), 1) \leq L \text{ and } v(t+1) = 0\right) \\
&= \mathbb{P}\left(v(t+1) = 0 \mid 0 < (v(t), 1) \leq L\right) \mathbb{P}\left(0 < (v(t), 1) \leq L\right).
\end{aligned}$$

By the death time estimate (3.6), we have

$$\mathbb{P}\left(v(t+1) = 0 \mid 0 < (v(t), 1) \leq L\right) \geq \exp\left(\frac{-\theta L}{1 - e^{-\theta}}\right)$$

so

$$\mathbb{P}(0 < (v(t), 1) \leq L) \leq \exp\left(\frac{\theta L}{1 - e^{-\theta}}\right) \mathbb{P}\left(\tau^{\phi} \in (t, t+1]\right).$$

Given any sequence $t_n \nearrow \infty$ such that $t_{n+1} - t_n \geq 1$ for each $n \in \mathbb{N}$,

$$\sum_{n \in \mathbb{N}} \mathbb{P}(0 < (v(t_n), 1) \leq L) \leq \exp\left(\frac{\theta L}{1 - e^{-\theta}}\right)$$

so it is clear that $\lim_{n\to\infty} \mathbb{P}(0 < (v(t_n), 1) \leq L) = 0$. This implies (3.20). To prove (3.21), we see that $\mathbb{P}\left(\tau^{\phi} \in (n, n+1]\right) \leq 1/n$ for infinitely many $n \in \mathbb{N}$, so we can choose a sequence $t_n \nearrow \infty$, $(t_n) \subset \mathbb{N}$, such that for each $n \in \mathbb{N}$,

$$\mathbb{P}(0 < (v(t_n), 1) \leq L) \leq t_n^{-1} \exp\left(\frac{\theta L}{1 - e^{-\theta}}\right).$$

We choose $c(\theta) = (1 - e^{-\theta})/2\theta$ and $L(t) = c \log t$. Then

$$\mathbb{P}(0 < (v(t_n), 1) \leq c \log t_n) \leq t_n^{-1/2}$$

which converges to zero as required. □

Lemma 3.6.3 *Let v and u_0 be as in the proof of Proposition 3.6.1, with $\theta > \theta_c$. Then for each fixed $R > 0$,*

$$\mathbb{P}\left((u_0, v(t_n)) \geq R \mid (v(t_n), 1) \geq c \log t_n\right) \to 1 \tag{3.22}$$

as $n \to \infty$, where c and t_n are as in Lemma 3.6.2. Furthermore,

$$\mathbb{P}(u_0 \equiv 0) = 0. \tag{3.23}$$

Proof For each $\delta > 0$ and $x \in \mathbb{R}$, we set H_x^{δ} to be the random variable

$$H_x^{\delta} = \inf_{y \in (x, x+\delta)} u_0(x) \wedge 1.$$

Note that the distribution of H_x^δ is independent of x by the translation invariance of u_0. Fix $\varepsilon > 0$. Since

$$\left|\mathbb{E}\left[H_0^\delta H_L^\delta\right] - \left(\mathbb{E}H_0^\delta\right)^2\right| \le 4\mathbb{E}\left[\sup_{x\in(-\delta,\delta)} |u_0(0) - u_0(x)|\right] + \left|\mathbb{E}\left[(u_0(0)\wedge 1)(u_0(L)\wedge 1)\right] - \mathbb{E}\left[u_0(0)\wedge 1\right]^2\right|$$

and

$$\mathbb{E}H_0^\delta \ge \mathbb{E}[u_0(0)\wedge 1] - \mathbb{E}\left[\sup_{x\in(-\delta,\delta)} |u_0(x) - u_0(0)|\right]$$

we can use the pointwise mixing hypothesis to find $I \in \mathbb{N}$ sufficiently large and $\delta \in (0,1]$ sufficiently small such that

$$\sup_{|x-y|\ge\delta I} \left|\mathbb{E}\left[H_x^\delta H_y^\delta\right] - \left(\mathbb{E}H_0^\delta\right)^2\right| < \varepsilon \tag{3.24}$$

and

$$\mathbb{E}H_0^\delta \ge \frac{1}{2}\mathbb{E}[u_0(0)\wedge 1] =: \rho > 0. \tag{3.25}$$

First, we calculate the probability of (u_0, f) being small, where f is a fixed nonzero deterministic function in $C_c^+(\mathbb{R})$. Note that

$$(f,1) = \sum_{i\in\delta\mathbb{Z}} (f, \chi_{(i,i+\delta)}) = \sum_{j=0}^{I-1} \sum_{i\in\delta I\mathbb{Z}+j} (f, \chi_{(i,i+\delta)})$$

so $\sum_{i\in\delta I\mathbb{Z}+j} (f, \chi_{(i,i+\delta)})$ must be at least $(f,1)/I$ for some $j \in \{0,\ldots,I-1\}$. We can therefore choose a finite set of points $\{x_i\}$ in $\delta\mathbb{Z}$ such that $|x_i - x_j| \ge \delta I$ for $i \ne j$ and

$$\sum_i (f, \chi_{(x_i,x_i+\delta)}) \ge \frac{(f,1)}{I}.$$

We set $y_i = (f, \chi_{(x_i,x_i+\delta)})$ and set $X = \sum y_i H_{x_i}^\delta$. Since $(u_0, f) \ge X$ and $\mathbb{E}X \ge (\rho/I)(f,1)$ we have

$$\mathbb{P}\left((u_0,f) \le \frac{\rho}{2I}(f,1)\right) \le \mathbb{P}\left(X \le \frac{1}{2}\mathbb{E}X\right) \le 4\left(\frac{\mathbb{E}[X^2] - (\mathbb{E}X)^2}{(\mathbb{E}X)^2}\right).$$

Now

$$
\begin{aligned}
(\mathbb{E}X)^2 &= (\mathbb{E}H_0^\delta)^2 \left(\sum y_i^2 + \sum_{i \neq j} y_i y_j \right) \\
\mathbb{E}\left[X^2\right] &= \mathbb{E}\left[(H_0^\delta)^2\right] \sum y_i^2 + \sum_{i \neq j} y_i y_j \mathbb{E}\left[H_{x_i}^\delta H_{x_j}^\delta\right]
\end{aligned}
$$

so, using (3.24) and (3.25),

$$
\frac{\mathbb{E}[X^2] - (\mathbb{E}X)^2}{(\mathbb{E}X)^2} \leq \frac{1}{\rho^2} \left(\varepsilon + \frac{\sum y_i^2}{\left(\sum y_i\right)^2} \right).
$$

To control the second term, note that $\sum y_i^2 / (\sum y_i)^2$ is always bounded by 1, and for any $K > 0$ it is also bounded by $K / \sum y_i$ if each $y_i \leq K$. To remove the dependence on our choice of $\{x_i\}$, we set $B_K^\delta(f)$ to be the event

$$
\left\{ (f, \chi_{(x, x+\delta)}) > K \text{ for some } x \in \delta\mathbb{Z} \right\}
$$

(which is nonrandom for deterministic f). We conclude that for any $\varepsilon > 0$ there exist $I \in \mathbb{N}$ and $\delta > 0$ such that for all $K > 0$ and $f \in C_c^+(\mathbb{R})$,

$$
\mathbb{P}\left((u_0, f) \leq \frac{\rho}{2I}(f, 1) \right) \leq \frac{4}{\rho^2} \left(\varepsilon + \chi_{B_K^\delta(f)} + \frac{KI}{(f, 1)} \right). \tag{3.26}
$$

We now need to plug in the random $v(t)$ in place of our deterministic f. For $\log t \geq 2IR/\rho c$,

$$
\begin{aligned}
&\mathbb{P}\left((u_0, v(t)) \leq R \mid (v(t), 1) \geq c \log t\right) \\
&\quad \leq \mathbb{P}\left((u_0, v(t)) \leq \frac{\rho}{2I}(v(t), 1) \mid (v(t), 1) \geq c \log t \right) \\
&\quad = \frac{\mathbb{P}\left((u_0, v(t)) \leq (\rho/2I)(v(t), 1) \text{ and } (v(t), 1) \geq c \log t\right)}{\mathbb{P}\left((v(t), 1) \geq c \log t\right)}.
\end{aligned}
$$

The denominator here is bounded below by $(1/2)\mathbb{P}\left(\tau^\phi = \infty\right)$ along t_n for sufficiently large n by the previous lemma. By independence of u_0 and $v(t)$, we see that, since $v(t)$ has distribution $\Theta_t\phi$,

$$\begin{aligned}
&\mathbb{P}\left((u_0, v(t)) \leq \frac{\rho}{2I}(v(t), 1) \text{ and } (v(t), 1) \geq c\log t\right) \\
&\quad= \int \mathbb{P}\left((u_0, f) \leq \frac{\rho}{2I}(f, 1)\right) \chi\{(f,1) \geq c\log t\} \, (\Theta_t\phi)(df) \\
&\quad\leq \int \frac{4}{\rho^2}\left(\varepsilon + \chi_{B^\delta_K(f)} + \frac{KI}{c\log t}\right) (\Theta_t\phi)(df) \\
&\quad= \frac{4}{\rho^2}\left(\varepsilon + \mathbb{P}\left(B^\delta_K(v(t))\right) + \frac{KI}{c\log t}\right).
\end{aligned}$$

To complete the proof of (3.22), we need to choose $K = K(t)$ such that, as $t \to \infty$,

$$\lim_{t\to\infty} \mathbb{P}\left(B^\delta_{K(t)}(v(t))\right) + \frac{K(t)}{\log t} = 0.$$

To control the probability term above, we control the behaviour of $v(t)$ on an interval which is linearly growing in t and use the wavespeed property (3.5) to assert that $v(t)$ is increasingly likely to be zero outside the interval, i.e.

$$\begin{aligned}
\mathbb{P}\left(B^\delta_{K(t)}(v(t))\right) &\leq \mathbb{P}(\operatorname{supp} v(t) \not\subseteq [-\alpha t, \alpha t]) \\
&\quad+ \left\lceil \frac{2\alpha t}{\delta} \right\rceil \sup_{x\in\mathbb{R}} \mathbb{P}\left((v(t), \chi_{(x,x+\delta)}) > K(t)\right).
\end{aligned}$$

By (3.5), we can choose $\alpha > 0$ so that the first term converges to zero. Finally,

$$t\mathbb{P}\left((v(t), I_{(x,x+\delta)}) > K(t)\right) \leq te^{-K(t)^{3/2}} \mathbb{E}\left[\exp\left((v(t), \chi_{(x,x+\delta)})^{3/2}\right)\right].$$

Since the expectation is bounded in x and t by Lemma 3.3.3 we can choose $K(t) = (\log t)^{3/4}$. This completes the proof of (3.22).

To prove (3.23), we refer back to (3.26) and note that if $f \leq 1$ and $K = 1$, $\chi_{B^\delta_K(f)}$ is always zero. So for each $\varepsilon > 0$ there exists $I \in \mathbb{N}$ such that for all

$f \in C_c^+(\mathbb{R})$ with $f \leq 1$,

$$\mathbb{P}\left(u_0 \equiv 0\right) \leq \mathbb{P}\left((u_0, f) \leq \frac{\rho}{2I}\right) \leq \frac{4}{\rho^2}\left(\varepsilon + \frac{I}{(f,1)}\right).$$

Now (3.23) easily follows. □

Proposition 3.6.4 *Suppose u is a solution of (3.1) with initial condition $f \in C_{tem}^+$, and $\inf_{x\in\mathbb{R}} f(x) \geq \delta$ for some $\delta > 0$. Then for each $\phi \in C_c^+(\mathbb{R})$,*

$$\lim_{t\to\infty} \mathbb{E}\left[e^{-(u(t),\phi)}\right] = \mathbb{P}\left(\tau^\phi < \infty\right).$$

In particular, the distribution of $u(t)$ converges weakly to the unique stationary distribution above.

Proof Let v be a solution of (3.1) with initial condition ϕ. Since $(v(t), 1) \leq R/\delta$ if $(v(t), f) \leq R$, we have

$$\begin{aligned}\mathbb{P}\left(\tau^\phi \leq t\right) \leq \mathbb{E}\left[e^{-(u(t),\phi)}\right] &= \mathbb{E}\left[e^{-(v(t),f)}\right] \\ &\leq \mathbb{P}\left(\tau^\phi \leq t\right) + \mathbb{P}\left(0 < (v(t), 1) \leq R/\delta\right) + e^{-R}.\end{aligned}$$

The result follows from the first part of Lemma 3.6.2. □

We can now prove that the distribution μ_∞^∞ constructed in Section 3.5 is non-trivial as claimed.

Corollary 3.6.5 *Suppose u is a solution of (3.1) with constant initial condition $N > 0$, and $\theta > \theta_c$. Then for each $x \in \mathbb{R}$,*

$$\liminf_{t\to\infty} \mathbb{E}\left[u(t,x)\right] > 0.$$

Proof Choose any $\phi \in C_c^+(\mathbb{R})$ such that $\phi \neq 0$. Since $x \geq 1 - e^{-x}$ and by Proposition 3.6.4,

$$\liminf_{t\to\infty} \mathbb{E}\left[(u(t), \phi)\right] \geq \lim_{t\to\infty} \mathbb{E}\left[1 - e^{-(u(t),\phi)}\right] = \mathbb{P}\left(\tau^\phi = \infty\right) > 0.$$

Because u is translation invariant, $\mathbb{E}\left[(u(t), \phi)\right] = \mathbb{E}\left[u(t,x)\right] \left(\int \phi(y)\, dy\right)$, so the result follows. $\square$

Appendix A

A brief introduction to white noise and martingale measures

Here, we briefly review the definitions and main ideas of space-time white noise and martingale measures from Walsh [32] needed for the thesis. The reader is referred to [32] for details.

If $(E, \mathcal{E}, \nu)$ is a σ-finite measure space, a *white noise* based on ν is a random set function $W : \mathcal{A} \times \Omega \to \mathbb{R}$ on the sets $\mathcal{A} = \{A \in \mathcal{E} \ : \ \nu(A) < \infty\}$ such that

1. $W(A)$ is a normal random variable with zero mean and variance $\nu(A)$,

2. if $A \cap B = \emptyset$ then $W(A)$ and $W(B)$ are independent and $W(A) + W(B) = W(A \cup B)$.

In this thesis, $(E, \mathcal{E}, \nu)$ will be $\mathbb{R}^+ \times \mathbb{R}$ equipped with the Borel σ-algebra and Lebesgue measure. In this case, white noise can be thought of as the analogue of Brownian motion for stochastic differential equations involving both space and time, and is often called the *Brownian sheet*.

Given a filtration $\{\mathcal{F}_t\}$, we say that a function $\phi : \mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ is *predictable* if it is in the σ-algebra generated by finite sums of functions of the form

$$f(x, s, \omega) = X(\omega)\chi_{(a,b]}(s)\chi_A(x),$$

where $0 \leq a < t$, X is bounded and $\mathcal{F}_a$-measurable and $A \in \mathcal{B}(\mathbb{R})$.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a complete filtered probability space and let W be a $\{\mathcal{F}_t\}$-adapted space-time white noise. For each $\{\mathcal{F}_t\}$-predictable function ϕ : $\mathbb{R}_+ \times \mathbb{R} \times \Omega \to \mathbb{R}$ such that $\int_0^t \int \phi(s, x, \omega)^2 \, dx \, ds < \infty$ for all $t > 0$, Walsh [32] constructs the stochastic integral $\int_0^t \int \phi(s, x, \omega) \, W(dx \, ds)$ as an $\{\mathcal{F}_t\}$-local martingale with quadratic variation

$$\left\langle \int_0 \int \phi(s, x, \omega) \, W(dx \, ds) \right\rangle_t = \int_0^t \int \phi(s, x, \omega)^2 \, dx \, ds.$$

More generally, Walsh [32] constructs the stochastic integral over what he calls a *martingale measure*, and defines this integral as a new martingale measure. In Section 1.5 we obtain martingale measures from the solutions of martingale problems, and use these to construct new martingale measures which we show to be white noises, so it will be useful to give a summary of this more general construction.

Let $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be the real line with the Borel σ-algebra, and let $\mathcal{A}$ be the set of sets in $\mathcal{B}(\mathbb{R})$ with finite Lebesgue measure. Suppose $U : \mathcal{A} \times \Omega \to \mathbb{R}$ is such that $\mathbb{E}[U(A)^2] < \infty$ for all $A \in \mathcal{A}$, and $U(A) + U(B) = U(A \cup B)$ for $A, B \in \mathcal{A}$. We say that U is *countably additive* if $\lim_{j\to\infty} \mathbb{E}[U(A_j \cap [-n, n])^2] = 0$ for any

$n \in \mathbb{N}$ and any sequence $\{A_j\}_{j\in\mathbb{N}} \subset \mathcal{A}$ such that $A_j \searrow \emptyset$. We call such a function a *σ-finite L^2-valued measure*.

Let $\{\mathcal{F}_t\}_{t\geq 0}$ be a right-continuous filtration. We say that $\{M_t(A)\}_{t\geq 0, A\in\mathcal{A}}$ is a *martingale measure* on $\mathbb{R}$ if

1. $M_0(A) = 0$ for all $A \in \mathcal{A}$,

2. $M_t(\cdot)$ is a σ-finite L^2-valued measure for all $t > 0$,

3. $\{M_t(A)\}_{t\geq 0}$ is an $\mathcal{F}_t$-martingale.

The main example we consider is white noise: set $W_t(A) = W((0,t] \times A)$. Then it is clear that $\{W_t(A)\}_{t\geq 0, A\in\mathcal{A}}$ is a martingale measure if we take $\mathcal{F}_t$ to be the filtration generated by W. Given a martingale measure M, we can define a set function Q_M on rectangles by

$$Q_M(A \times B \times (s,t]) = \langle M(A), M(B)\rangle_t - \langle M(A), M(B)\rangle_s$$

and extend by additivity to finite disjoint unions of rectangles.

A martingale measure is *orthogonal* if $\langle M(A), M(B)\rangle_t = 0$ for disjoint A and B, i.e. Q_M only has mass on the diagonal. We only consider orthogonal measures in this thesis, although Walsh [32] deals with a more general class of martingale measures which he calls *worthy*. In particular, the fact that white noise is orthogonal is immediate from the independence of $W(A)$ and $W(B)$ on disjoint sets. By Proposition 2.10 of Walsh [32], an orthogonal martingale measure M such that $t \mapsto M_t(A)$ is continuous is a white noise if and only if Q_M is deterministic. Such an M is a standard white noise if $Q_M(dx\,dy\,ds) = \delta_{\{x=y\}}\,dx\,dy\,ds$, i.e. $M_t(A)$ has variance given by $t|A|$.

Given a predictable function f and an orthogonal martingale measure M, such that, for $A \in \mathcal{A}$,

$$\mathbb{E}\left[\int_0^t \int_A \int_A f(x,s)f(y,s)Q_M(dx\,dy\,ds)\right] < \infty,$$

Walsh constructs a martingale measure $f \bullet M$ with covariance measure

$$Q_{f\bullet M}(dx\,dy\,ds) = f(x,s)f(y,s)\,Q_M(dx\,dy\,ds).$$

Here, the notation $f \bullet M_t(A)$ corresponds to the usual integral notation, i.e.

$$f \bullet M_t(A) = \int_0^t \int_A f(s,x)\,M(dx\,ds)$$

are two ways of expressing the same thing. Since we have defined the stochastic integral in terms of another martingale measure, we can repeat the integration over $f \bullet M$. In particular, we show in Section 1.5 how to construct white noise as the stochastic integral of a martingale measure and prove that a certain process satisfies (1.1) with respect to this white noise.

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